SOME RECENT DEVELOPMENTS IN THE THEORY OF CONTINUED FRACTIONS[1]

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The principal rôle of the continued fraction in analysis has perhaps been that of an intermediary between more familiar and easily handled things, such as between the power series and the integral. This may partly explain the fact that there are certain questions about continued fractions which have remained relatively unexplored. To illustrate what I mean, if one's principal attention were focused upon power series, and continued fractions were used only incidentally, it is unlikely that one would imagine that the convergence region for the continued fraction is, as it now appears, more properly a *parabolic* region than a circular region.

I wish to speak today about some results which have been obtained during the last few years, by a group of men with whom I have been associated. Our investigations have been centered mainly upon the continued fraction itself. In certain instances it has been possible to apply our results to problems not directly connected with continued fractions. Thus, during the course of this lecture I shall have occasion to speak of the problem of moments, of Hausdorff summability, and of certain classes of analytic functions.

1. **Some definitions and formulas.** Before discussing some of the problems with which we have been concerned, I shall put down some necessary definitions and formulas. The continued fractions considered are chiefly of the form

$$\frac{1}{1}+\frac{a_2}{1}+\frac{a_3}{1}+\frac{a_4}{1}+\cdots \tag{1.1}$$

in which $a_2, a_3, a_4, \cdots$ are complex numbers. Apart from unessential initial irregularities, any continued fraction in which the partial denominators are different from zero can be thrown into this form. The nth *approximant* of (1.1) is the ordinary fraction, A_n/B_n, obtained by stopping with the nth partial quotient. The *numerators* and *denominators* may be computed by means of the recursion formulas

$$\begin{aligned} &A_0 = 0, \quad B_0 = 1, \quad A_1 = 1, \quad B_1 = 1, \\ &A_n = A_{n-1} + a_n A_{n-2}, \quad B_n = B_{n-1} + a_n B_{n-2}, \quad n = 2, 3, 4, \cdots . \end{aligned} \tag{1.2}$$

[1] An address delivered before the Detroit meeting of the Society, November 23, 1940, by invitation of the Program Committee.

An important consequence of the recursion formulas is the determinant formula

$$A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}a_2a_3 \cdots a_n. \tag{1.3}$$

To assign a meaning to the continued fraction, we suppose, first, that $B_n \neq 0$ from and after some n. If, then, the limit $\lim_{n=\infty} (A_n/B_n)$ exists and is finite, the continued fraction is said to converge, and the value of this limit is the *value* of the continued fraction. The convergence problem is the problem of determining conditions upon $a_2, a_3, a_4, \cdots$ which are sufficient for the convergence of the continued fraction. For instance, one may determine a "convergence region" in which $a_2, a_3, a_4, \cdots$ may vary independently and the continued fraction remains convergent. Some idea of the nature of this problem may be had from the fact that it is possible to choose $a_2, a_3, a_4, \cdots$ in such a way that these numbers form an everywhere dense set in the complex plane, and the continued fraction is convergent.[2] Worpitzky[3] showed that (1.1) converges if $|a_n| \leqq 1/4$. This is the best circular convergence region with center at the origin, inasmuch as (1.1) diverges if $a_n = -\frac{1}{4} - c$, $c > 0$. Szász[4] found that if a is not a real number $\leqq -\frac{1}{4}$, then there exists a positive number r such that (1.1) converges if $|a_n - a| \leqq r$. We[5] found recently that if $R(a) > -1/4$ r may be taken equal to $\frac{1}{4}(|1+2a| - 2|a|)$. Again, if $|a_{2n+1}| \leqq 1/4$, $|a_{2n}| \geqq 25/4$, (1.1) is convergent.[6]

There is one important *necessary* condition for convergence due to von Koch,[7] which applies whenever the partial numerators are different from zero. Write the continued fraction in the form

$$\frac{1}{1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4} + \cdots \tag{1.4}$$

where $a_n = 1/b_{n-1}b_n$, $(n = 2, 3, 4, \cdots, b_1 = 1)$. Then, if the series $\sum |b_n|$ *converges*, the sequences of even and odd approximants have distinct

[2] Walter Leighton and H. S. Wall, *On the transformation and convergence of continued fractions* (to be referred to as T), American Journal of Mathematics, vol. 58 (1936), pp. 267–281, p. 269.

[3] Worpitzky, Jahresbericht, Friedrichs-Gymnasium und Realschule, Berlin, 1865, pp. 3–39. Independent proofs of this theorem were given later by Pringsheim and by Van Vleck.

[4] O. Perron, *Die Lehre von den Kettenbrüchen* (to be referred to as "Perron"), 2d edition, Leipzig and Berlin, 1929, p. 282.

[5] W. T. Scott and H. S. Wall, *A convergence theorem for continued fractions* (to be referred to as CT), Transactions of this Society, vol. 47 (1940), pp. 155–172, p. 171.

[6] T, p. 278.

[7] Perron, pp. 235–236.

limits (one may be infinite), and the continued fraction diverges by oscillation.

This natural breaking up of the sequence of approximants into two sequences makes it often of importance to consider the continued fraction

$$(1.5)\qquad \frac{1}{1+a_2}-\frac{a_2a_3}{1+a_3+a_4}-\frac{a_4a_5}{1+a_5+a_6}-\cdots$$

having A_{2n}/B_{2n} as its nth approximant; and the continued fraction

$$(1.6)\quad 1-\frac{a_2}{1+a_2+a_3}-\frac{a_3a_4}{1+a_4+a_5}-\frac{a_5a_6}{1+a_6+a_7}-\cdots,$$

having A_{2n+1}/B_{2n+1} as its nth approximant.[8] We shall refer to these as the *even part* and the *odd part*, respectively, of (1.1).

2. **Some ideas on convergence.** A natural procedure to follow in attacking the convergence problem is to turn to the infinite series

$$\frac{A_1}{B_1}+\left(\frac{A_2}{B_2}-\frac{A_1}{B_1}\right)+\left(\frac{A_3}{B_3}-\frac{A_2}{B_2}\right)+\cdots,$$

which, by (1.3), may be written in the form

$$(2.1)\qquad 1-\frac{a_2}{B_1B_2}+\frac{a_2a_3}{B_2B_3}-\frac{a_2a_3a_4}{B_3B_4}+\cdots.$$

The starting point in our investigation is the idea of forming a majorant for the series (2.1) by requiring that the test-ratio

$$\rho_n=\frac{(-1)^n a_2a_3\cdots a_{n+1}}{B_nB_{n+1}}:\frac{(-1)^{n-1}a_2a_3\cdots a_n}{B_{n-1}B_n}=\frac{-a_{n+1}B_{n-1}}{B_{n+1}}$$

remain numerically less than or equal to some number r_n:

$$(2.2)\qquad |\rho_n|\leqq r_n,\qquad n=1,2,3,\cdots.$$

If this holds, then it is necessarily true that $B_n\neq 0$, $(n=1, 2, 3, \cdots)$. Hence the continued fraction converges if some a_n vanishes, or, in any case, if the majorant series $1+\sum r_1r_2\cdots r_n$ converges. Moreover, the sum of this majorant series is an upper bound for the absolute value of the continued fraction. If the a_n's are variables, and the r_n's are independent of the a_n's when the latter lie in a certain domain,

[8] Perron, p. 201.

then the continued fraction converges uniformly over this domain if the majorant series converges.

Now, in order to translate the condition (2.2) into a condition upon $a_2, a_3, a_4, \cdots$ we have the simple relation

$$\rho_n = \frac{-a_{n+1}(1+\rho_{n-1})}{1+a_{n+1}(1+\rho_{n-1})}, \qquad n = 1, 2, 3, \cdots, \rho_0 = 0. \tag{2.3}$$

Using this, the desired translations may be made in a great many different ways. We have found[9] the following to be especially useful:

$$r_n|1+a_n+a_{n+1}| \geqq r_n r_{n-2}|a_n| + |a_{n+1}|, \tag{2.4}$$

$(n=1, 2, 3, \cdots)$, where we agree to put $r_{-1}=r_0=a_1=0$. If, for example, we take $r_n=n/(n+2)$, so that the majorant series $1+\sum r_1 r_2 \cdots r_n=2$, we find at once the theorem of Worpitsky mentioned in §1, namely: (1.1) converges (uniformly) for $|a_n| \leqq 1/4$, $(n=2, 3, 4, \cdots)$. The upper bound 2 which we find in this case is the least upper bound inasmuch as the value of (1.1) is 2 when $a_n=-1/4$.

Again, by making an appropriate choice of the r_n's, one can derive the theorem that the continued fraction

$$\frac{g_1}{1+}\ \frac{(1-g_1)g_2x_2}{1+}\ \frac{(1-g_2)g_3x_3}{1+}\ \cdots \tag{2.5}$$

in which $0<g_1<1$, $0 \leqq g_n<1$, $(n=2, 3, 4, \cdots)$, converges uniformly for $|x_n| \leqq 1$, $(n=2, 3, 4, \cdots)$, and its absolute value in this domain does not exceed

$$1-\frac{1}{1+\sum_{n=1}^{\infty}\frac{g_1g_2\cdots g_n}{(1-g_1)(1-g_2)\cdots(1-g_n)}}. \tag{2.6}$$

This value is actually attained by the continued fraction if $x_n=-1$. We thus improve in important respects the well known *Pringsheim criteria*. An immediate consequence is the theorem that

$$\frac{1}{1+}\ \frac{g_1x_1}{1+}\ \frac{(1-g_1)g_2x_2}{1+}\ \frac{(1-g_2)g_3x_3}{1+}\ \cdots \tag{2.7}$$

converges uniformly for $|x_n| \leqq 1$, $(n=1, 2, 3, \cdots)$, if the series in (2.6) converges; and for $|x_1| \leqq h<1$, $|x_n| \leqq 1$, $(n=2, 3, 4, \cdots)$, if this same series diverges or converges. Part of this theorem was

[9] CT, p. 155 ff.

proved by E. B. Van Vleck,[10] who seems to have been the first to recognize the importance of continued fractions of this form.

The relationship of the inequalities (2.4) to continued fractions of the form (2.7) is indicated by the following theorem.

THEOREM 2.1. *If the inequalities* (2.4) *hold with actual inequality for* $n=1, 2,$ *and if* $a_n \neq 0$, $(n=2, 3, 4, \cdots)$, *then the even and odd parts of* (1.1) *can, except for initial irregularities, be written in the form* (2.7). *In fact, the even part can be put into the form*

$$(2.8)\quad \frac{1}{1+a_2}\left\{\frac{1}{1}+\frac{g_1x_1}{1}+\frac{(1-g_1)g_2x_2}{1}+\frac{(1-g_2)g_3x_3}{1}+\cdots\right\}$$

where $|x_1|<1$, $|x_n|\leqq 1$, $(n=2, 3, 4, \cdots)$, $0<g_n<1$, $(n=1, 2, 3, \cdots)$; *and the odd part can be put into the form*

$$(2.9)\quad 1-\frac{a_2}{1+a_2+a_3}\left\{\frac{1}{1}+\frac{h_1y_1}{1}+\frac{(1-h_1)h_2y_2}{1}+\frac{(1-h_2)h_3y_3}{1}+\cdots\right\},$$

where $|y_1|<1$, $|y_n|\leqq 1$, $(n=2, 3, 4, \cdots)$, $0<h_n<1$, $(n=1, 2, 3, \cdots)$.

Put

$$g_n=\frac{r_{2n+1}|1+a_{2n+1}+a_{2n+2}|-|a_{2n+2}|}{r_{2n+1}|1+a_{2n+1}+a_{2n+2}|},$$

$$h_n=\frac{r_{2n+2}|1+a_{2n+2}+a_{2n+3}|-|a_{2n+3}|}{r_{2n+2}|1+a_{2n+2}+a_{2n+3}|}.$$

It follows from the inequalities (2.4) that

$$g_1\geqq\frac{r_1|a_3|}{|1+a_3+a_4|},\qquad h_1\geqq\frac{r_2|a_4|}{|1+a_4+a_5|},$$

$$(1-g_{n-1})g_n\geqq\left|\frac{a_{2n}a_{2n+1}}{(1+a_{2n-1}+a_{2n})(1+a_{2n+1}+a_{2n+2})}\right|,$$

$$(1-h_{n-1})h_n\geqq\left|\frac{a_{2n+1}a_{2n+2}}{(1+a_{2n}+a_{2n+1})(1+a_{2n+2}+a_{2n+3})}\right|.$$

If then we put

$$\frac{-a_3a_4}{(1+a_2+a_3)(1+a_4+a_5)}=h_1y_1,\qquad \frac{-a_2a_3}{(1+a_2)(1+a_3+a_4)}=g_1x_1,$$

[10] E. B. Van Vleck, Transactions of this Society, vol. 2 (1901), pp. 476–483.

$$\frac{-a_{2n+1}a_{2n+2}}{(1+a_{2n}+a_{2n+1})(1+a_{2n+2}+a_{2n+3})} = (1-h_{n-1})h_n y_n,$$

$$\frac{-a_{2n}a_{2n+1}}{(1+a_{2n-1}+a_{2n})(1+a_{2n+1}+a_{2n+2})} = (1-g_{n-1})g_n x_n,$$

we see on referring to (1.5), (1.6) that the theorem is true.

Now, both the continued fractions (2.8), (2.9) are convergent. Hence, to prove (1.1) convergent it is required to show that the values of (2.8) and (2.9) are equal. For that purpose we find that[11]

$$\left|\frac{A_n}{B_n}-\frac{A_{n+1}}{B_{n+1}}\right| = \frac{1}{|Q_nQ_{n+1}|}$$

where

$$r_1r_3\cdots r_{2n+1}|Q_{2n+2}| \geqq e_1\left(1+\sum_{k=1}^{n} r_1r_3^2\cdots r_{2k-1}^2 r_{2k+1}|b_{2k+2}|\right),$$

$$r_2r_4\cdots r_{2n+2}|Q_{2n+3}| \geqq e_2\left(1+\sum_{k=1}^{n+1} r_2r_4^2\cdots r_{2k-2}^2 r_{2k}|b_{2k+1}|\right),$$

e_1, e_2 being positive constants, $b_1=1$, $a_n=1/b_{n-1}b_n$ (cf. (1.4)). It is now easy to see that (1.1) converges, under certain conditions, even if the majorant series $1+\sum r_1r_2\cdots r_n$ diverges. It suffices, for example, to have simply $\liminf (r_1r_2\cdots r_n)=0$ provided actual inequality holds for $n=1, 2$ in (2.4). Other interesting cases are cited in the following theorem.

THEOREM 2.2. *If* (2.4) *holds with actual inequality for* $n=1, 2$, *then* (1.1) *converges if any one of the following conditions holds*:

(a) $|a_n|<M$, $n=2, 3, 4, \cdots$, (*M a positive constant*);

(b) $\liminf |a_n|=0$;

(c) $\liminf (r_1r_2\cdots r_n)<\infty$, $\sum(1/|a_n|)$ *diverges*;

(d) $r_1r_3\cdots r_{2n-1}<M$, $r_2r_4\cdots r_{2n}<M$ (*M a positive constant*), $\sum|b_n|$ *diverges*; *or*

(e) $\liminf (r_1r_2\cdots r_n)<\infty$, $r_1r_3\cdots r_{2n-1}$ *and* $r_2r_4\cdots r_{2n}$ *bounded away from zero*, $\sum|b_n|$ *diverges*.

3. **The parabola theorem.** As noted in §1, the domain $|a_n|\leqq 1/4$ is the best possible circular convergence region with center at the origin. Perhaps the most interesting result of our investigations of the convergence problem is the theorem which follows.[12]

[11] CT, p. 164.

[12] CT, p. 166.

THEOREM 3.1. *A set of points W which is symmetrical with respect to the real axis is a convergence region*[13] *for the continued fraction* (1.1) *if and only if W is bounded, and is contained in or upon the parabola*

$$|z| - R(z) = \tfrac{1}{2}. \tag{3.1}$$

The necessity of the conditions follows from the fact that if z lies outside the parabola (3.1), then

$$\frac{1}{1} + \frac{z}{1} + \frac{\bar{z}}{1} + \frac{z}{1} + \frac{\bar{z}}{1} + \cdots \tag{3.2}$$

diverges; and from the fact that a convergence region must be bounded, for otherwise $a_2, a_3, a_4, \cdots$ could be chosen in the region in such a way that the series $\sum|b_n|$ of (1.4) would be convergent. The sufficiency follows immediately from Theorem 2.2 (a), since it is easy to verify that the inequalities (2.4) hold with $r_n = 1$, and with actual inequality for $n = 1, 2$, when the a_n's lie in or upon the parabola (3.1).

From Theorem 2.2 (d) we have further that if the a_n's lie in or upon the parabola[14] (3.1), then (1.1) converges if (and only if) the series $\sum|b_n|$ diverges.

No less important and interesting than the convergence region problem is the companion value region problem:[15] if U is a convergence region, to determine a value region V in which the value of (1.1) must lie when $a_2, a_3, a_4, \cdots$ lie in U. If it is known that

(a) u in U implies that $1/(1+u)$ is in V;

(b) u in U, v in V implies that $1/(1+uv)$ is in V,

then it is plain that the approximants A_n/B_n, $(n = 2, 3, 4, \cdots)$, all lie in V whenever $a_2, a_3, a_4, \cdots$ lie in U. By employing this interpretation of the continued fraction as a succession of linear transformations we have succeeded in determining the exact region V in which the approximants of (1.1) must lie when $a_2, a_3, a_4, \cdots$ lie in the parabola (3.1). We find that V is the region

$$|z - 1| \leqq 1, \qquad z \neq 0. \tag{3.3}$$

Every value z satisfying (3.3) is assumed by some continued fraction of the form (1.1) having its partial numerators in the parabola (3.1): and no value z outside this region is so assumed.

[13] I.e. the continued fraction converges when $a_2, a_3, a_4, \cdots$ have arbitrary values in W.

[14] Not necessarily a bounded portion of the parabola.

[15] W. T. Scott and H. S. Wall, *Value regions for continued fractions* (to be referred to as VR), to be published in this Bulletin.

We have also investigated the value region problem corresponding to a circular convergence region with center at an arbitrary point a not real and $\leqq -1/4$.

It is interesting to consider the following "restricted" value region problem. Suppose that $a_2, a_3, \cdots, a_n$ are fixed in U. To determine a value region V in which the approximants must lie when $a_{n+1}, a_{n+2}, a_{n+3}, \cdots$ vary independently in U. We have considered this problem for the case that U is the region bounded by the parabola (3.1). To illustrate the sort of result obtained, let $a_2=\frac{1}{2}$, $a_3=\frac{1}{6}$, $a_4=\frac{1}{3}$, so that the continued fraction starts out like the simple continued fraction for log 2. Then if (1.1) converges, and $a_5, a_6, a_7, \cdots$ lie in the parabola (3.1), the value of the continued fraction must lie in a circle with center[16] $z=.69375$ $(\log 2=.69315\cdots)$ and radius .00625.

It is interesting to apply these considerations to special continued fractions. For instance, one may show in this way that the hypergeometric function $F(\alpha, 1, \gamma, x)$, $(\alpha, \gamma$, real, $0<\alpha<\gamma)$, satisfies the inequality[17]

$$\left| F(\alpha, 1, \gamma+1, x) - \frac{\gamma^2}{\gamma^2-\alpha^2} \right| \leqq \frac{\alpha\gamma}{\gamma^2-\alpha^2}, \qquad |x| \leqq 1.$$

4. **The moment problem[18] for the interval** (0, 1). We shall return now to the continued fraction (2.7), in which we shall put $x_n=x$. We shall also multiply by a positive constant factor c_0. We then have:

$$(4.1) \qquad \frac{c_0}{1+}\ \frac{g_1x}{1+}\ \frac{(1-g_1)g_2x}{1+}\ \frac{(1-g_2)g_3x}{1+}\ \cdots,$$

[16] Formulas for the center and radius in terms of a_2, a_3, a_4 are given in VR.

[17] H. S. Wall, *A class of functions bounded in the unit circle*, Duke Mathematical Journal, vol. 7 (1940), pp. 146–153.

[18] By the "moment problem" is ordinarily understood the problem of determining a monotone nondecreasing function $\phi(u)$ satisfying the equations $c_n=\int_a^b u^n d\phi(u)$, $(n=0, 1, 2, \cdots)$, when $\{c_n\}$ is a given sequence. This problem was solved by Stieltjes for the case $a=0$, $b=+\infty$ in 1894 (*Oeuvres*, vol. 2, pp. 402–566). In 1903 E. B. Van Vleck (*On an extension of the 1894 memoir of Stieltjes*, Transactions of this Society, vol. 4 (1903), pp. 297–332) investigated the problem for the case $a=-\infty$, $b=+\infty$. He did not obtain a complete solution. H. Hamburger (Mathematische Annalen, vol. 81 (1920), pp. 235–319, and vol. 82 (1921), pp. 120–164, 168–187) obtained a complete solution in the general case. At about the same time, E. Hellinger (Mathematische Annalen, vol. 86 (1922), pp. 18–29) gave a complete solution in a twelve-page article, by resolving the problem into a question of solving a system of infinitely many linear equations. Another solution was given by T. Carleman, *Sur les équations intégrales singulières à noyau symétrique*, Uppsala, 1923, pp. 189–220.

with $c_0>0$, $0<g_n<1$, $(n=1, 2, 3, \cdots)$. This is a *Stieltjes* type continued fraction inasmuch as the constant factors in the partial numerators are all positive. Since this continued fraction converges uniformly for $|x| \leqq c$ for every positive $c<1$, it follows that the corresponding Stieltjes integral has the form

$$\int_0^1 \frac{d\phi(u)}{1+xu}, \tag{4.2}$$

in which $\phi(u)$ is bounded and monotone nondecreasing and has infinitely many points of increase, $0 \leqq u \leqq 1$. Hence, if $P(x)=\sum c_n(-x)^n$ is the corresponding Stieltjes series, it follows that the sequence $\{c_n\}$ is a *totally monotone sequence*, i.e.,

$$\Delta^m c_n = \int_0^1 (1-u)^m u^n d\phi(u) \geqq 0, \qquad m, n = 0, 1, 2, \cdots .$$

Conversely, let $\{c_n\}$ be a totally monotone sequence. Then, as Hausdorff showed, there exists a function $\phi(u)$ such that

$$c_n = \int_0^1 u^n d\phi(u), \qquad n = 0, 1, 2, \cdots ,$$

where $\phi(u)$ is bounded and monotone nondecreasing. If $\phi(u)$ has infinitely many points of increase ("there is an infinite distribution of mass"), the series $P(x)=\sum c_n(-x)^n$ has a corresponding Stieltjes continued fraction of the form

$$\cfrac{c_0}{1+\cfrac{a_2x}{1+\cfrac{a_3x}{1+}}} \cdots , \tag{4.3}$$

in which $a_2, a_3, a_4, \cdots$ are real and positive. Then one may show[19] that (4.3) must be of the form (4.1). Thus, in order for the real sequence $\{c_n\}$ to be totally monotone and correspond to an infinite distribution of mass it is necessary and sufficient that the corresponding Stieltjes continued fraction be of the form (4.1).

As for totally monotone sequences corresponding to a finite distribution of mass, they are completely characterized by having a corresponding continued fraction of the form (4.1) which terminates. In this case the last g_n may be 0 or 1.

The Stieltjes moment problem for the interval (0, 1), for which we have completely characterized the corresponding continued fraction,

[19] H. S. Wall, *Continued fractions and totally monotone sequences* (to be referred to as TM), Transactions of this Society, vol. 48 (1940), pp. 165–184.

is of particular importance on account of its relation to Hausdorff methods of summation. I shall discuss some of our work on this theory in a later section.

5. **A class of real functions bounded in the unit circle.** Let $f(x)$ denote the moment generating function represented by (4.1). Then $f(x)$ has an integral representation of the form (4.2), and a power series expansion $f(x)=\sum c_n(-x)^n$ in which the sequence of coefficients, $\{c_n\}$, is totally monotone. We shall include in our discussion the case where (4.1) terminates, in which event the last g_n may be 0 or 1.

Let $M(f)=\text{l.u.b.}_{|x|<1}|f(x)|$. Then $M(f)\leqq 1$ if and only if the continued fraction can be put into the form[20]

$$(5.1)\qquad \frac{g_1}{1+}\ \frac{(1-g_1)g_2x}{1}\ +\ \frac{(1-g_2)g_3x}{1}\ +\ \cdots$$

where $0\leqq g_n\leqq 1$, $(n=1, 2, 3, \cdots)$, and we agree that in case equality holds here for some n, then the continued fraction shall terminate with the first identically vanishing partial quotient. Again, $M(f)\leqq 1$ if and only if $\sum c_n\leqq 1$; or, if and only if

$$f(x)=\int_0^1 \frac{(1-u)d\phi(u)}{1+xu},$$

where $\phi(u)$ is monotone and $0\leqq\phi(1)-\phi(0)\leqq 1$.

With each function $e(x)$, analytic for $|x|<1$, and with $M(e)\leqq 1$, Schur[21] associated a sequence of numbers $\{\gamma_n\}$ in the following way. Put $e(x)\equiv e_0(x)$,

$$(5.2)\qquad e_{n+1}(x)=\frac{1}{x}\,\frac{\gamma_n-e_n(x)}{1-\bar{\gamma}_n e_n(x)},\quad \gamma_n=e_n(0),\qquad n=0, 1, 2, \cdots.$$

He showed that either (a) $|\gamma_n|<1$, $n=0, 1, 2, \cdots$; or else (b) $|\gamma_n|<1$, $n=0, 1, 2, \cdots, N-1$, $|\gamma_N|=1$, $\gamma_n=0$, $n>N$. Conversely, if $\{\gamma_n\}$ is any sequence satisfying (a) or (b) then there is uniquely determined a function $e(x)$, analytic for $|x|<1$ and with $M(e)\leqq 1$, such that (5.2) holds.

We shall confine our attention to the class E of *real* functions $e(x)$, analytic for $|x|<1$ and with $M(e)\leqq 1$ In this case the sequence $\{\gamma_n\}$ associated with $e(x)$ in accordance with the result of Schur satisfies

[20] TM, p. 179. Naturally the g_n's are not the same in (5.1), (5.2).

[21] I. Schur, *Ueber Potenzreihen, die im Innern des Einheitskreises beschränkt sind*, Journal für die reine und angewandte Mathematik, vol. 147 (1916), pp. 205–232, and vol. 148 (1917), pp. 122–145.

the condition (a) $-1<\gamma_n<+1$, $n=0, 1, 2, \cdots$; or else the condition (b) $-1<\gamma_n<+1$, $n=0, 1, 2, \cdots, N-1$, $|\gamma_N|=1$, $\gamma_n=0$, $n>N$. The subclass consisting of all moment generating functions of E will be denoted by F. If $e(x)$ is in F, then all the functions $e_1(x), e_2(x), e_3(x), \cdots$ given by (5.2) are also in F.

We shall now prove the theorem which follows.

THEOREM 5.1. *There is a one-to-one correspondence between the functions of E and of F such that if e(x) of E corresponds to f(z) of F, then for* $|x|<1$ *we have*

$$\frac{(1-x)}{2}\,\frac{1-e(x)}{1+xe(x)}=f(z), \qquad z=4x/(1-x)^2. \tag{5.3}$$

This may be formulated in terms of monotone functions: to every function $e(x)$ of E there corresponds a monotone function $\phi(u)$ such that $0\leqq\phi(1)-\phi(0)\leqq 1$ and

$$\frac{1-x}{x}\,\frac{1-e(x)}{1+xe(x)}=\int_0^1\frac{(1-u)d\phi(u)}{1+zu}, \qquad z=4x/(1-x)^2, \quad |x|<1. \tag{5.4}$$

Conversely, if $\phi(u)$ is any monotone function such that $0\leqq\phi(1)-\phi(0)\leqq 1$, then there exists a function $e(x)$ of E such that (5.4) holds.

Let $e_0(x)\equiv e(x)$, $e_1(x), e_2(x), \cdots$ be determined by (5.2), and put

$$p_n(x)=\frac{1-e_n(x)}{1+xe_n(x)}, \qquad n=0, 1, 2, \cdots.$$

Then from (5.2) we have the relation

$$p_n(x)=\frac{1-\gamma_n}{1-x+(1+\gamma_n)xp_{n+1}(x)}, \qquad n=0, 1, 2, \cdots,$$

and consequently there is the *formal* continued fraction expansion

$$\tfrac{1}{2}(1-x)\,\frac{1-e(x)}{1+xe(x)}=\frac{g_1}{1}+\frac{(1-g_1)g_2z}{1}+\frac{(1-g_2)g_3z}{1}+\cdots, \tag{5.5}$$

where $z=4x/(1-x)^2$, $g_n=\frac{1}{2}(1-\gamma_{n-1})$, $(n=1, 2, 3, \cdots)$. If some γ_n is $+1$ or -1, this continued fraction terminates and (5.5) is then an identity. If, on the other hand, $-1<\gamma_{n-1}<+1$, then $0<g_n<1$, and the continued fraction converges uniformly for $|z|\leqq 1$, or for x in a sufficiently small neighborhood of the origin. One may easily show that the power series expansion in ascending powers of x of the func-

tion on the left agrees term-by-term with the power series expansion in ascending powers of x of the nth approximant of the continued fraction, for more and more terms as n is increased. From these facts one may conclude[22] that (5.5) is a true equality for x in a sufficiently small neighborhood of the origin. Now,[23] the function $z=4x/(1-x)^2$ maps the interior of the circle $|x|=1$ in a one-to-one manner upon the z-plane exterior to the cut along the real axis from -1 to $-\infty$. The continued fraction represents an analytic function of z in this cut plane, and therefore represents an analytic function of x for $|x|<1$. Thus (5.5) is a true equality for $|x|<1$.

As previously pointed out, this continued fraction represents a function $f(z)$ of the class F.

Conversely, starting with the function $f(z)$ of F, so that the numbers g_n may be found, we then determine the sequence $\{\gamma_n\}$ and hence the function $e(x)$ of E, by putting $\gamma_{n-1}=1-2g_n$, $(n=1, 2, 3, \cdots)$. If $0<g_n<1$, then $-1<\gamma_n<+1$, and the g_n's and γ_n's are uniquely determined. In case the continued fraction for $f(z)$ terminates, we may always assume that the last g_n is either 0 or 1, and take subsequent g_n's all equal to $\frac{1}{2}$. In this case we determine $e(x)$ as before by taking $\gamma_{n-1}=1-2g_n$. The function $e(x)$ clearly satisfies (5.5), and the theorem is proved.

Denote by $q(x)$ the function in the left member of (5.5). Then we find that

$$\frac{1-q(x)}{1+zq(x)}=\tfrac{1}{2}(1-x)\frac{1+e(x)}{1-xe(x)}.$$

Now if we replace $e(x)$ by $-e(x)$ in (5.5), it is clear that the effect upon the continued fraction on the right is to replace γ_{n-1} by $-\gamma_{n-1}$, which is the same as replacing g_n by $1-g_n$. Hence we have the theorem[24] that if

$$p(x)=\frac{g_1}{1}+\frac{(1-g_1)g_2x}{1}+\frac{(1-g_2)g_3x}{1}+\cdots,$$

then the continued fraction for $[1-p(x)]/[1+xp(x)]$ is obtained

[22] Cf. the argument in Perron, p. 343.

[23] Cf. E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, Berlin, 1929, p. 112.

[24] This theorem also follows from Theorem 2.1, p. 166, of TM. This was used in the paper referred to in Footnotes 14, 24 and also in the paper by H. S. Wall: *A continued fraction related to some partition formulas of Euler*, American Mathematical Monthly, vol. 48 (1941), pp. 102–108.

from that for $p(x)$ by replacing g_n by $1-g_n$, $(n=1, 2, 3, \cdots)$. It follows that if $p(x)$ is in the class F, so is $[1-p(x)]/[1+xp(x)]$. Therefore we have, in consequence of Theorem 5.1, the following theorem:

THEOREM 5.2. *To each function $p(x)$ of the class F there corresponds uniquely a function $f(z)$ of F such that*

$$(5.6) \qquad (1-x)p(x) = 2f(z), \qquad z = 4x/(1-x)^2, \ |x| < 1.$$

From Theorem 5.1 we obtain the following result:

THEOREM 5.3. *Let $e(x)$ be a function of E such that $-1<\gamma_n<+1$, $(n=0, 1, 2, \cdots)$. Let the series*

$$S = 1 + \sum_{n=0}^{\infty} \frac{(1-\gamma_0)(1-\gamma_1)\cdots(1-\gamma_n)}{(1+\gamma_0)(1+\gamma_1)\cdots(1+\gamma_n)}$$

be convergent. Then $M(e)=1$.

It is interesting to compare this with the result of Schur which states that if the series $\sum|\gamma_n|$ converges, then $M(e)<1$.

To prove the theorem, let us recall that the series S is the series appearing in (2.6) with $g_n=\frac{1}{2}(1-\gamma_{n-1})$. Hence, when this series converges we conclude by Theorem 5.1 that

$$|q(x)| = \left|\frac{1-x}{2}\,\frac{1-e(x)}{1+xe(x)}\right| \leqq (1-1/S) < 1$$

if $|4x/(1-x)^2|<1$. Let x be real, and let $x\to-1$, $|x|<1$. Then $|4x/(1-x)^2|<1$ and $4x/(1-x)^2\to-1$, so that $q(x)\to1-(1/S)$. It follows that $e(x)\to1$, so that $M(e)=1$.

Inasmuch as $M[e(x)]=M[-e(x)]=M[e(-x)]=M[-e(-x)]$, it follows that the convergence of one of the series obtained from S by replacing γ_n by $-\gamma_n$, γ_{2n+1} by $-\gamma_{2n+1}$, or γ_{2n} by $-\gamma_{2n}$, $(n=0, 1, 2, \cdots)$, is sufficient in order that $M(e)=1$.

By a result of Stieltjes, the function $f(z)$ represented by the continued fraction on the right of (5.5) is a meromorphic function of z if and only if $\lim_{n=\infty}(1-g_n)g_{n+1}=0$, that is:

$$\lim_{n=\infty}(1+\gamma_{n-1})(1-\gamma_n) = 0.$$

In this case we may write

$$q(x) = \frac{(1-x)[1-e(x)]}{2[1+xe(x)]} = (1-x)^2\sum_{n=1}^{\infty}\frac{M_n}{(x-r_n)(x-\bar{r}_n)}$$

where $M_n>0$, $(n=1, 2, 3, \cdots)$, $\sum M_n$ converges, and $r_n=1-2u_n+2i[u_n(1-u_n)]^{1/2}$, $1>u_1>u_2>\cdots>0$, $\lim u_n=0$. Thus the singularities of $q(x)$ consist of simple poles r_n, $\bar{r}_n$ lying upon the circle $|x|=1$, together with the point $x=1$ which is the unique limit point of these poles. The function $q(x)$ satisfies the relation $q(1/x)=q(x)$. The following theorem is now evident.

THEOREM 5.4. *If $e(x)$ is a function of E such that $-1<\gamma_n<+1$, $\lim_{n=\infty}(1+\gamma_{n-1})(1-\gamma_n)=0$, then $e(x)$ has an essential singularity at $x=1$, and no other singularities in the extended plane except poles. The function*

$$q(x)=\frac{1-x}{2}\,\frac{1-e(x)}{1+xe(x)}$$

has as its singularities an infinite sequence of simple poles lying on the circle $|x|=1$ with the single limit point $x=1$, and no others; and $q(1/x)=q(x)$.

As an example, if

$$e(x)=-\frac{1}{x}\,\frac{\dfrac{2x^{1/2}}{1+x}-\tanh\dfrac{2x^{1/2}}{1-x}}{\dfrac{2x^{1/2}}{1+x}+\tanh\dfrac{2x^{1/2}}{1-x}},$$

then $q(x)=3^{-1}/1+3^{-1}5^{-1}z/1+5^{-1}7^{-1}z/1+\cdots$, $z=4x/(1-x)^2$, as one may verify by means of the continued fraction of Lambert for $\tanh z$. This function $e(x)$ satisfies the conditions of Theorem 5.4. When $z=-1$ in the continued fraction for $q(x)$, we find that its value is $1-(\tan 1)^{-1}<1$, and consequently the series S of Theorem 5.3 converges. From that theorem it then follows that $M(e)=1$. This can be verified directly by letting x approach -1 in the above expression for $e(x)$.

Theorem 5.1 admits of the following interpretation. In (5.3) put $[1-e(x)]/[1+xe(x)]=c_0-c_1x+c_2x^2-\cdots$, and $f(z)=C_0-C_1z+C_2z^2-\cdots$. Considered as an identity in x, (5.3) yields the equations[25]

$$(5.7)\qquad c_n=2^{2n+2}C_{2n,0}C_n-2^{2n+1}C_{2n-1,1}C_{n-1}+\cdots+(-1)^n2^2C_{n,n}C_0,$$

which constitute a linear transformation of the sequence $\{C_n\}$ into

[25] W. T. Scott and H. S. Wall, *Linear manifolds of Hausdorff means* (to be referred to as LM). To appear in the Transactions of this Society as Part I of a paper: *The transformation of series and sequences.*

the sequence $\{c_n\}$. The inverse transformation is found to be

$$C_n = \frac{1}{(n+1)2^{2n+1}} C_{2n,n}c_0 + \frac{3}{(n+2)2^{2n+1}} C_{2n,n-1}c_1 + \cdots + \frac{2n+1}{(2n+1)2^{2n+1}} C_{2n,0}c_n. \tag{5.8}$$

The transformation (5.8) has the property that it carries any sequence $\{c_n\}$ generated by a function $[1-e(x)]/[1+xe(x)]$ where $e(x)$ is in E, into a totally monotone sequence $\{C_n\}$ with sum $\sum C_n \leqq 1$; and (5.7) has the property that it carries any totally monotone sequence $\{C_n\}$ with sum $\leqq 1$ into a sequence $\{c_n\}$ which is generated by a function of the form $[1-e(x)]/[1+xe(x)]$ where $e(x)$ is in E.

This leads us to the topic to be discussed in the next section.

6. **Hausdorff means.** The familiar $(C, 1)$ transform of a sequence $\{s_n\}$, namely: $t_m = (s_0 + s_1 + \cdots + s_m)/(m+1)$, may be written in the form

$$t_m = \sum_{n=0}^{m} C_{m,n} \int_0^1 (1-u)^{m-n} u^n s_n du.$$

One may generalize this by replacing du by $d\phi(u)$ where $\phi(u)$ is any function of bounded variation on the interval $0 \leqq u \leqq 1$, which is continuous at $u=0$, and which satisfies the conditions $\phi(0)=0$, $\phi(1)=1$. The resulting transform of the sequence $\{s_n\}$ is the Hausdorff mean.[26] This defines a regular method of summation, i.e., $s_n \to s$ implies $t_m \to s$, which is denoted by the symbol $[H, \phi(u)]$.

If we put $c_n = \int_0^1 u^n d\phi(u)$, $(n = 0, 1, 2, \cdots)$, the Hausdorff mean can be written in the form

$$t_m = \sum_{n=0}^{m} C_{m,n} \Delta^{m-n} c_n \cdot s_n.$$

The function $\phi(u)$, subject to the conditions imposed above, is called a *regular mass function*; and the sequence $\{c_n\}$ is called a *regular moment sequence*. Included among the Hausdorff means are the Hölder means (H, α), the Cesàro means (C, α), and many others.

Let $[H, \phi_a(u)]$ and $[H, \phi_b(u)]$ be two Hausdorff methods of summation, and let $\{a_n\}$ and $\{b_n\}$ be the corresponding regular moment se-

[26] F. Hausdorff, *Summationsmethoden und Momentfolgen*, I and II, Mathematische Zeitschrift, vol. 9 (1921), pp. 74–109, 280–299. For an elementary account, see H. L. Garabedian, *Hausdorff matrices*, American Mathematical Monthly, vol. 46 (1939), pp. 390–410.

quences. Then[27] $[H, \phi_a(u)] \supset [H, \phi_b(u)]$, when $b_n \neq 0$, $(n = 0, 1, 2, \cdots)$, if and only if there exists a regular moment sequence $\{c_n\}$ such that

$$a_n = b_n c_n, \qquad n = 0, 1, 2, \cdots. \tag{6.1}$$

This is the basic theorem of the Hausdorff theory. When (6.1) holds it is convenient to say that $\{a_n\}$ is *divisible* by $\{b_n\}$. The condition that $\{a_n\}$ be divisible by $\{b_n\}$ may be formulated in a number of different ways.[28] One of the most natural ways is in terms of the moment generating functions

$$f_a(x) = \sum a_n(-x)^n, \qquad f_b(x) = \sum b_n(-x)^n.$$

The regular moment sequence $\{a_n\}$ is divisible by the regular moment sequence $\{b_n\}$ if and only if there exists a regular mass function $\phi_c(u)$ such that

$$f_a(x) = \int_0^1 f_b(ux)d\phi_c(u)$$

for all x not real and $\leqq -1$. By means of the Stieltjes inversion formula one may show that this holds if and only if

$$\phi_a(u) = \phi_c(u) + \int_u^1 \phi_b(u/v)d\phi_c(v)$$

for all except at most a countable set of values of u.

If $\phi(u)$ is monotone nondecreasing, then $\{c_n\}$ is a totally monotone sequence, and hence the moment generating function $f_c(x)$ has a corresponding continued fraction of the form (4.1). It is easy to obtain conditions on (4.1) which are necessary and sufficient for $\{c_n\}$ to be a regular moment sequence.[29] An interesting example is afforded by the continued fraction of Gauss for the hypergeometric function $F(\alpha, 1, \gamma, x)$, α, γ real, $0 < \alpha < \gamma$. We have investigated in some detail the "hypergeometric summability" defined in this way.

It is interesting that in some cases one may operate directly with the continued fraction (4.1) to prove inclusion relationships between Hausdorff methods.[30]

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[27] This means that every sequence summable $[H, \phi_b(u)]$ is summable $[H, \phi_a(u)]$, and is read "$[H, \phi_a(u)]$ includes $[H, \phi_b(u)]$."

[28] H. L. Garabedian, Einar Hille, and H. S. Wall, *Formulations of the Hausdorff inclusion problem*, Duke Mathematical Journal, vol. 8 (1941), pp. 193–213.

[29] H. L. Garabedian and H. S. Wall, *Hausdorff methods of summation and continued fractions*, Transactions of this Society, vol. 48 (1940), pp. 185–207, p. 188.

[30] H. L. Garabedian and H. S. Wall, *Continued fractions and Hausdorff methods of summation*, Northwestern University Studies, in press.

Beginning with the interpretation of Theorem 5.1 given at the end of §5, we have developed a theory of linear manifolds of Hausdorff means.[31] Returning to (5.8), let us put

$$\beta_n(u) = \frac{1}{2^{2n+1}}\left\{\frac{1}{n+1}C_{2n,n} + \frac{3}{n+2}C_{2n,n-1}u + \cdots + \frac{2n+1}{2n+1}C_{2n,0}u^n\right\}.$$

Then if $c_n = \int_0^1 u^n d\phi(u)$, we may write the transformation (5.8) as

$$C_n = \int_0^1 \beta_n(u)d\phi(u), \qquad n = 0, 1, 2, \cdots. \tag{6.2}$$

From the discussion in §5 it follows that if $\{c_n\}$ is totally monotone and[32] $\sum c_n \leqq 1$, the sequence $\{C_n\}$ is of like character. This is, however, only part of the story. We find that if $\phi(u)$ is any function of bounded variation on the interval $0 \leqq u \leqq 1$, such that $\phi(1) - \phi(0) = 1$, then the sequence $\{C_n\}$ is a regular moment sequence.

Let $M(\beta_n(u)) = M(\beta)$ denote the set of all moment sequences $\{C_n\}$ obtained from (6.2) by letting $\phi(u)$ run through the class of all functions of bounded variation on the interval $0 \leqq u \leqq 1$. It is observed that if $\{C_n'\}$ and $\{C_n''\}$ are any two sequences in $M(\beta)$ then the sequences $\{C_n' + C_n''\}$, and $\{KC_n'\}$, (K a constant), are in $M(\beta)$. Thus $M(\beta)$ is a *linear manifold* of moment sequences.

Any suitably chosen sequence of functions $\{\beta_n(u)\}$ determines in this same way a linear manifold of moment sequences. We have called $\{\beta_n(u)\}$ the *basis* of the manifold. In laying down the outline for a general theory of these manifolds, we have obtained conditions on a sequence of functions $\{\beta_n(u)\}$ in order that it form the basis of a manifold; have obtained conditions under which the sequences of the manifold are all essentially regular; and conditions under which the Hausdorff methods defined by the sequences of a manifold all include a given Hausdorff method.

As an example, let $\beta_n(u) = (u+1)/(u+n+1)$. For every fixed u in the interval $0 \leqq u \leqq 1$, the sequence $\{\beta_n(u)\}$ is totally monotone. From this one may conclude that $\{\beta_n(u)\}$ is a basis for a manifold $M(\beta)$. Further, it can be shown that every sequence $\{C_n\}$ in $M(\beta)$ for which $C_0 \neq 0$ is essentially regular; and that every Hausdorff method of summation defined by the sequences of $M(\beta)$ includes $(C, 1)$ but not all of them include $(C, 1+t)$, $t > 0$.

7. **Continued fraction expansions for arbitrary power series.** The

[31] LM.

[32] If $f(x)$ is in F and $f(x) = \sum c_n(-x)^n$, then $M(f) = \sum c_n$. Cf. TM, p. 181.

continued fraction expansions for power series which we have used up to this point all have partial quotients of the form $a_n x/1$. This is true only when the power series is of a special kind. Any power series $P(x)=1+c_1x+c_2x^2+\cdots$ has a corresponding continued fraction in which the partial quotients are of the form $a_n x^{\alpha_n}/1$, α_n a positive integer.[33] The continued fraction has many of the properties of semi-normal continued fractions. For example, it terminates if and only if $P(x)$ represents a rational function of x; and if the continued fraction converges uniformly in the vicinity of the origin, the power series also converges in the neighborhood of the origin and the power series is equal to its continued fraction.

We have investigated these "corresponding type" continued fractions with a view toward obtaining formulas for the a_n's and α_n's in terms of the power series.[34]

We found first of all that there is a fairly practical step-by-step process for expanding $P(x)$ into a continued fraction. This is based upon the observation that if we have carried out the expansion to the point where we have

$$1+\frac{a_1x^{\alpha_1}}{1}+\frac{a_2x^{\alpha_2}}{1}+\cdots+\frac{a_nx^{\alpha_n}}{1},$$

and we denote this fraction by $A_n(x)/B_n(x)$, then the formal power series for

$$\frac{A_n(x)}{B_n(x)}-P(x) \tag{7.1}$$

must begin with the term $(-1)^{n+1}a_1a_2\cdots a_{n+1}x^{\alpha_1+\alpha_2+\cdots+\alpha_{n+1}}$. Hence, knowing $a_1, a_2, \cdots, a_n, \alpha_1, \alpha_2, \cdots, \alpha_n$ we may determine a_{n+1} and α_{n+1} as soon as we know the term of lowest degree in the power series for (7.1). This process requires expansions in powers of x for only a *finite* number of terms at each stage.

We have found formulas for the elements of the continued fraction in terms of the power series in the case of a large class of continued fractions which we have termed *absolutely regular*, namely those for which $\alpha_1=1$ and

[33] Walter Leighton and W. T. Scott, *A general continued fraction expansion*, Bulletin of this Society, vol. 45 (1939), pp. 596–605.

[34] W. T. Scott and H. S. Wall, *Continued fraction expansions for arbitrary power series*, Annals of Mathematics, (2), vol. 41 (1940), pp. 328–349.

$$\begin{aligned} \alpha_2 + \alpha_4 + \cdots + \alpha_{2n} &\geqq \alpha_1 + \alpha_3 + \cdots + \alpha_{2n-1}, \\ \alpha_3 + \alpha_5 + \cdots + \alpha_{2n+1} &\geqq \alpha_2 + \alpha_4 + \cdots + \alpha_{2n}, \end{aligned} \qquad n = 1, 2, 3, \cdots .$$

In case the a_n's are real and positive, these conditions are equivalent to the condition that all the approximants of the continued fraction be Padé approximants for the corresponding power series. The class of absolutely regular continued fractions contains the class of semi-normal continued fractions, and the formulas found are natural extensions of the well known formulas for the semi-normal case.

A remarkable situation arises when each exponent in the power series $P(x)$ is at least twice the preceding. In this case the a_n's in the continued fraction depend only upon the coefficients in the power series, while the α_n's in the continued fraction depend only upon the exponents in the power series. For this reason, certain operations upon the power series and continued fraction, such as differentiation, integration, forming reciprocals, and a sort of Hadamard composition, may be readily performed.

There is considerable evidence to support the conjecture that corresponding type continued fractions represent functions having circles as natural boundaries, provided the exponents α_n increase sufficiently rapidly. We have found that this is the case when the a_n's are suitably restricted and the α_n's form a geometric progression.

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ON BIORTHOGONAL MATRICES[1]

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Introduction. Consider the basis consisting of a number system $\mathfrak{A}$ of type D, two general ranges $\mathfrak{P}^1$, $\mathfrak{P}^2$, and two positive hermitian matrices ϵ^1, ϵ^2. We introduce two binary relations for pairs of non-modular matrices. The matrices κ^{12}, ϕ^{21} are said to be contraceding as to $\epsilon^1\epsilon_\kappa^2$, ϵ^2 in case κ^{12}, ϕ^{21} are by columns of $\mathfrak{M}(\epsilon^1)$, $\mathfrak{M}(\epsilon^2)$ respectively and such that $J^{2\kappa}\kappa^{12}\mu^2 = J^2\phi^{*12}\mu^2$ for every μ^2 in the set $\mathfrak{M}(\epsilon_\kappa^2 \cap \epsilon^2)$. It is evident that when κ^{12} is of type $\mathfrak{M}(\epsilon^1)\mathfrak{M}(\epsilon^2)$, then the contracedence property implies that $J^2\kappa^{12}\phi^{21} = \epsilon_\kappa^1$ but not conversely. The main results are stated in Theorems 2 and 3. We next consider ϵ_0^1, ϵ_1^1 both idempotent as to ϵ^1. Suppose that κ^{12} is by columns of $\mathfrak{M}(\epsilon_0^1)$ and ϕ^{21} is by rows-conjugate of $\mathfrak{M}(\epsilon_1^1)$. Take any pair of vectors μ^1, ν^1 modular as to ϵ_0^1, ϵ_1^1 respectively such that $J^1\kappa^{*21}\mu^1$, $J^1\phi^{21}\nu^1$ are in $\mathfrak{M}(\epsilon^2)$. If $J^1\bar{\mu}^1\nu^1$ is equal to the inner product $J^2(J^1\bar{\mu}^1\kappa^{12}, J^1\phi^{21}\nu^1)$, then κ^{12}, ϕ^{21} are said to be biorthogonal as to $\epsilon^1\epsilon_0^1\epsilon_1^1\epsilon^2$. When $\epsilon_1^1 = \epsilon_0^1$, then κ^{12}, ϕ^{21} are said to be biorthogonal as to $\epsilon^1\epsilon_0^1\epsilon^2$ in case they are biorthogonal as to $\epsilon^1\epsilon_0^1\epsilon_0^1\epsilon^2$. With proper restrictions imposed upon κ^{12}, ϕ^{21}, we obtain the contracedence property. In a later paper, we shall establish the relations of biorthogonality and a certain mode of interchange of integration processes.

1. **Preliminary results.** Consider the basis $\mathfrak{A}$, $\mathfrak{P}^1$, $\mathfrak{P}^2$, ϵ^1, and κ^{12} which is by columns of $\mathfrak{M}(\epsilon^1)$. E. H. Moore's generalized Fourier processes give $\epsilon_\kappa^2 \equiv J^1\kappa^{*21}\kappa^{12}$ and $\epsilon_\kappa^1 \equiv J^{2\kappa}\kappa^{12}\kappa^{*21}$. The spaces $\mathfrak{M}(\epsilon_\kappa^1)$ and $\mathfrak{M}(\epsilon_\kappa^2)$ are in one-to-one correspondence (denoted by $\leftrightarrow$) via the transformations $J^1\kappa^{*21}$ and $J^{2\kappa}\kappa^{12}$, and the correspondences are orthogonal in the sense that the moduli of the corresponding vectors are preserved.[2]

(A)[3] Suppose that $\mathfrak{M}_1(\epsilon_\kappa^1) \leftrightarrow \mathfrak{M}_1(\epsilon_\kappa^2)$ and $\mathfrak{M}_2(\epsilon_\kappa^1) \leftrightarrow \mathfrak{M}_2(\epsilon_\kappa^2)$ via the transformations $J^1\kappa^{*21}$, $J^{2\kappa}\kappa^{12}$. Then $\mathfrak{M}_1(\epsilon_\kappa^1)$ is a subset of $\mathfrak{M}_2(\epsilon_\kappa^1)$ if and only if $\mathfrak{M}_1(\epsilon_\kappa^2)$ is a subset of $\mathfrak{M}_2(\epsilon_\kappa^2)$; $\mathfrak{M}_1(\epsilon_\kappa^1)$ is linearly J^1-closed if and only if $\mathfrak{M}_1(\epsilon_\kappa^2)$ is linearly $J^{2\kappa}$-closed; and $\mathfrak{M}_1(\epsilon_\kappa^1)$ is everywhere dense in $\mathfrak{M}_2(\epsilon_\kappa^1)$ if and only if $\mathfrak{M}_1(\epsilon_\kappa^2)$ is everywhere dense in $\mathfrak{M}_2(\epsilon_\kappa^2)$.

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[1] Presented to the Society, June 20, 1940.

[2] For a concise outline of Moore's generalized Fourier theory and its related topics, *see* Moore, *General Analysis*, I, pp. 19–26. For an important classical instance, *see* E. Schmidt, *Über die Auflösung linearer Gleichungen mit unendlichvielen Unbekannten*, Rendiconti del Circolo Matematico di Palermo, vol. 25 (1908), pp. 56–77.

[3] For the demonstrations of the following results, *see* the author's forthcoming paper *On non-modular matrices*.

Let ϵ^2 be a positive hermitian matrix. (B) The class of all vectors μ^1 modular as to ϵ^1 such that $J^1\kappa^{*21}\mu^1$ is in $\mathfrak{M}(\epsilon^2)$ will be denoted by $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$. The intersection of $\mathfrak{M}(\epsilon^2)$ and $\mathfrak{M}(\epsilon_\kappa^2)$ will be denoted by $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$. (C) The set $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$ is identical with the class of vectors $J^1\kappa^{*21}\mu^1$ for all μ^1 in $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$. Moreover, (D) the sets $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ and $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$ are in one-to-one (orthogonal) correspondence via the transformations $J^1\kappa^{*21}$ and $J^{2\kappa}\kappa^{12}$. (E) The set $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$ is the linear extension of the sum of $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ and the orthogonal complement within $\mathfrak{M}(\epsilon^1)$ of $\mathfrak{M}(\epsilon_\kappa^1)$. (F) The class $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$ is everywhere dense in $\mathfrak{M}(\epsilon^1)$ if and only if $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$ is everywhere dense in $\mathfrak{M}(\epsilon_\kappa^2)$.

2. **Contraceding pairs of matrices.** The basis of the paper consists of a number system $\mathfrak{A}$ of type D, two general ranges $\mathfrak{P}^1$, $\mathfrak{P}^2$, and two positive hermitian matrices ϵ^1, ϵ^2.

LEMMA 1. *Suppose that κ^{12} is by columns of $\mathfrak{M}(\epsilon^1)$. Let $\mathfrak{M}_0(\epsilon^1)$ be a subset of $\mathfrak{M}(\epsilon^1)$; then $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ is contained in (contains) $\mathfrak{M}_0(\epsilon^1)$ only if $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$ is contained in (contains) the class of vectors $J^1\kappa^{*21}\mu^1$ for all μ^1 in $\mathfrak{M}_0(\epsilon^1)$. The converses are valid provided that $\mathfrak{M}_0(\epsilon^1)\subset\mathfrak{M}(\epsilon_\kappa^1)$.*

PROOF. The lemma follows from (D) and (A) in §1.

If ϕ^{21} is by columns of $\mathfrak{M}(\epsilon^2)$, we introduce the notations due to E. H. Moore:

$$\overset{1}{\epsilon}_\phi \equiv J^2\phi^{*12}\phi^{21}, \qquad \overset{2}{\epsilon}_\phi \equiv J^1\phi^{21}\phi^{*12}.$$

LEMMA 2. *Suppose that κ^{12}, ϕ^{21} are by columns of $\mathfrak{M}(\epsilon^1)$, $\mathfrak{M}(\epsilon^2)$ respectively. Then $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ is contained in (contains) $\mathfrak{M}(\epsilon_\kappa^1\cap\epsilon_\phi^1)$ if and only if $\mathfrak{M}(\epsilon_\kappa^2\cap\epsilon^2)$ is contained in (contains) the class of vectors $J^1\kappa^{*21}\mu^1$ for all μ^1 in $\mathfrak{M}(\epsilon_\kappa^1\cap\epsilon_\phi^1)$.*

PROOF. This lemma is a special instance of Lemma 1.

DEFINITION 1. *The matrices κ^{12}, ϕ^{21} are said to be contraceding as to $\epsilon^1\epsilon_\kappa^2\epsilon^2$ in case κ^{12}, ϕ^{21} are by columns of $\mathfrak{M}(\epsilon^1)$, $\mathfrak{M}(\epsilon^2)$ respectively and such that $J^{2\kappa}\kappa^{12}\mu^2 = J^2\phi^{*12}\mu^2$ for every μ^2 in the set $\mathfrak{M}(\epsilon_\kappa^2\cap\epsilon^2)$.*

THEOREM 1. *Suppose that κ^{12}, ϕ^{21} are contraceding as to $\epsilon^1\epsilon_\kappa^2\epsilon^2$. Then*
(1) $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2) = [J^2\phi^{*12}\mu^2 \,|\, \mu^2 \text{ in } \mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)] \subset \mathfrak{M}(\epsilon_\kappa^1\cap\epsilon_\phi^1)$;
(2) $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)\subset\mathfrak{M}(\epsilon^2\phi\epsilon_\kappa^1)$;
(3) $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)\subset\mathfrak{M}(\epsilon^1\cap\epsilon_\phi^1)$ *if and only if every μ^1 for which $J^1\epsilon_\kappa^1\mu^1 = 0^1$ is in $\mathfrak{M}(\epsilon_\phi^1)$.*

PROOF. Part (1) is a consequence of the fact that

$$\mathfrak{M}(\overset{1}{\epsilon}_\kappa\overset{*}{\kappa}\overset{2}{\epsilon}) = [J^{2\kappa}\overset{12}{\kappa}\overset{2}{\mu} \,|\, \overset{2}{\mu} \text{ in } \mathfrak{M}(\overset{2}{\epsilon}\cap\overset{2}{\epsilon}_\kappa)].$$

By hypothesis, we may replace $J^{2\kappa}\kappa^{12}$ by $J^2\phi^{*12}$. Since ϕ^{*12} is modular as to $\epsilon_\phi^1\epsilon^2$ and $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ is a subset of $\mathfrak{M}(\epsilon_\kappa^1)$, every vector of the form $J^2\phi^{*12}\mu^2$ must belong to $\mathfrak{M}(\epsilon_\kappa^1\cap\epsilon_\phi^1)$. This proves (1). Part (2) is obvious from (1). Part (3) follows from (1) and (E).

THEOREM 2. *Suppose that* ϕ^{21}, κ^{12} *are contraceding as to* $\epsilon^2\epsilon_\phi^1\epsilon^2$. *Then*

(1) $O^1\mathfrak{M}(\epsilon_\kappa^1)\subset\mathfrak{M}(\epsilon_\phi^1)$ *if and only if* κ^{12} *is complete by columns of* $\mathfrak{M}(\epsilon^1)$;

(2) *the following four assertions are equivalent*: (i) $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)\subset\mathfrak{M}(\epsilon_\phi^1)$; (ii) κ^{12} *is complete by columns of* $\mathfrak{M}(\epsilon^1)$ *and* $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)\subset\mathfrak{M}(\epsilon_\phi^2\phi^*\epsilon^1)$; (iii) κ^{12} *is complete by columns of* $\mathfrak{M}(\epsilon^1)$ *and* κ^{12}, ϕ^{21} *are contraceding as to* $\epsilon^1\epsilon_\kappa^2\epsilon^2$; (iv) $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)\subset[J^2\phi^{*12}\mu^2\,|\,\mu^2$ *in* $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)]$.

If one of the four conditions in (2) *is valid, then* $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)=\mathfrak{M}(\epsilon^1\cap\epsilon_\phi^1)$ *and* $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)=\mathfrak{M}(\epsilon_\phi^2\phi^*\epsilon^1)$.

PROOF. If κ^{12} is complete by columns of $\mathfrak{M}(\epsilon^1)$, then $O^1\mathfrak{M}(\epsilon_\kappa^1)=[0^1]$, which is, of course, contained in $\mathfrak{M}(\epsilon_\phi^1)$. Conversely, every μ^1 satisfying $J^1\bar{\mu}^1\kappa^{12}=0^2$ is in the orthogonal complement within $\mathfrak{M}(\epsilon^1)$ of $\mathfrak{M}(\epsilon_\kappa^1)$, and hence is in $\mathfrak{M}(\epsilon_\phi^1)$. Thus $J^{1\phi}\phi^{21}\mu^1=0^2$. Since ϕ^{21} is complete by rows-conjugate of $\mathfrak{M}(\epsilon_\phi^1)$, we have $\mu^1=0^1$, proving that κ^{12} is complete by columns of $\mathfrak{M}(\epsilon^1)$.

To prove (i) implies (ii), we observe by (E) that $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$ is the linear extension of the sum of $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ and $O^1\mathfrak{M}(\epsilon_\kappa^1)$. If (i) is valid, then by part (1) just proved, κ^{12} is complete by columns of $\mathfrak{M}(\epsilon^1)$. By Lemma 2, the condition that $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)\subset\mathfrak{M}(\epsilon_\phi^1)$ is equivalent to

$$\text{(a)}\qquad \mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)\subset[J^1\kappa^{*21}\mu^1\,|\,\mu^1\text{ in }\mathfrak{M}(\epsilon^1\cap\epsilon_\phi^1)].$$

By hypothesis, we may replace $J^1\kappa^{*21}$ by $J^{1\phi}\phi^{21}$. Since, by (D),

$$\text{(b)}\qquad [J^{1\phi}\phi^{21}\mu^1\,|\,\mu^1\text{ in }\mathfrak{M}(\epsilon^1\cap\epsilon_\phi^1)]=\mathfrak{M}(\epsilon_\phi^2\phi^*\epsilon^1),$$

we have (i)→(ii). To prove (ii)→(iii), consider any vector μ^2 in $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$. If (ii) is valid, then, by (D) (cf. (b) above), there exists a vector μ^1 in $\mathfrak{M}(\epsilon^1\cap\epsilon_\phi^1)$ such that

$$\text{(c)}\qquad \mu^2=J^{1\phi}\phi^{21}\mu^1,$$

whence

$$J^2\phi^{*12}\mu^2=J^2\phi^{*12}J^{1\phi}\phi^{21}\mu^1=J^{1\phi}\epsilon_\phi^1\mu^1=\mu^1.$$

Now by hypothesis and (c), we have $\mu^2=J^1\kappa^{*21}\mu^1$. As $\epsilon_\kappa^1=\epsilon^1$, it follows that

$$J^{2\kappa}\kappa^{12}\mu^2=J^{2\kappa}\kappa^{12}J^1\kappa^{*21}\mu^1=J^1\epsilon_\kappa^1\mu^1=\mu^1,$$

proving the condition (iii). By Theorem 1, we have (iii)→(iv). Since ϕ^{*12} is modular as to $\epsilon^1_\phi\epsilon^2$, we secure (iv)→(i).

The final statement follows by Theorem 1.

THEOREM 3. *The matrices ϕ^{21}, κ^{12} are contraceding as to $\epsilon^2\epsilon^1_\phi\epsilon^1$, ϕ^{21} is complete by columns of $\mathfrak{M}(\epsilon^2)$, and $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)\subset\mathfrak{M}(\epsilon^1_\phi)$ if and only if κ^{12}, ϕ^{21} are contraceding as to $\epsilon^1\epsilon^2_\kappa\epsilon^2$, κ^{12} is complete by columns of $\mathfrak{M}(\epsilon^1)$, and $\mathfrak{M}(\epsilon^2\phi^*\epsilon^1)\subset\mathfrak{M}(\epsilon^2_\kappa)$.*

PROOF. Apply Theorems 1 and 2.

3. **Biorthogonal matrices.** By considering $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2)$ and $\mathfrak{M}(\epsilon^2_\kappa\bigcap\epsilon^2)$ as subsets of $\mathfrak{M}(\epsilon^1_\kappa)$ and $\mathfrak{M}(\epsilon^2_\kappa)$ respectively, we have established the one-to-one correspondence between those two subsets. The correspondences are orthogonal. But the Fourier coefficient function of every vector in $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2)$ is also modular as to ϵ^2. This property gives rise to another direction of studying the correspondences between the aforementioned subsets.

DEFINITION 2. *Suppose that ϵ^1_0, ϵ^1_1 are idempotent*[4] *as to ϵ^1, κ^{12} is by columns of $\mathfrak{M}(\epsilon^1_0)$, and ϕ^{21} is by rows-conjugate of $\mathfrak{M}(\epsilon^1_1)$. Then κ^{12}, ϕ^{21} are said to be biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^1_1\epsilon^2$ in case*

$$J^1\mu^1\nu^1 = J^2(J^1\mu^1\kappa^{12}, J^1\phi^{21}\nu^1)$$

for every pair of vectors μ^1, ν^1 in $\mathfrak{M}(\epsilon^1_0\kappa^*\epsilon^2)$, $\mathfrak{M}(\epsilon^1_1\phi\epsilon^2)$ respectively.

It is obvious that κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^1_1\epsilon^2$ if and only if ϕ^{*12}, κ^{*21} are biorthogonal as to $\epsilon^1\epsilon^1_1\epsilon^1_0\epsilon^2$.

DEFINITION 3. *Suppose that ϵ^1_0 is idempotent as to ϵ^1 and κ^{12}, ϕ^{*12} are by columns of $\mathfrak{M}(\epsilon^1_0)$. Then κ^{12}, ϕ^{21} are said to be biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^2$ if they are biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^1_0\epsilon^2$.*

THEOREM 4. *Suppose that ϵ^1_0, ϵ^1_1 are idempotent as to ϵ^1, and κ^{12}, ϕ^{*12} are by columns of $\mathfrak{M}(\epsilon^1_0)$, $\mathfrak{M}(\epsilon^1_1)$ respectively. Then*

(i) *κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_\kappa\epsilon^1_1\epsilon^2$ if κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^1_1\epsilon^2$, and only if κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_\kappa\epsilon^1_{\phi^*}\epsilon^2$;*

(ii) *κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_\kappa\epsilon^1_1\epsilon^2$ if and only if*

$$J^2(J^1\mu^1\phi^{*12}, \mu^2) = J^{2*}(J^1\mu^1\kappa^{12}, \mu^2) \tag{4.1}$$

holds for every μ^1 in $\mathfrak{M}(\epsilon^1_1\phi\epsilon^2)$ and every μ^2 in $\mathfrak{M}(\epsilon^2\bigcap\epsilon^2_\kappa)$.

(iii) *κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_\kappa\epsilon_{\phi^*}\epsilon^2$ if and only if*

[4] ϵ^1_0 is idempotent as to ϵ^1 in case ϵ^1_0 is by columns of $\mathfrak{M}(\epsilon^1)$ and $J^1\epsilon^{1*}_0\epsilon^1_0=\epsilon_0$. See G.A., I, pp. 23–24.

(4.2) $$J^2\xi^2\eta^2 = J^1(J^{2\phi^*}\bar{\xi}^2\phi^{21}, J^{2\epsilon_\kappa}\kappa^{12}\eta^2)$$

holds for every ξ^2 *in* $\mathfrak{M}(\epsilon^2\cap\epsilon^2_{\phi^*})$ *and every* η^2 *in* $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$.

PROOF. Since $\mathfrak{M}(\epsilon^1_\kappa)$ and $\mathfrak{M}(\epsilon^1_{\phi^*})$ are subsets of $\mathfrak{M}(\epsilon^1_0)$ and $\mathfrak{M}(\epsilon^1_1)$ respectively, we have the conclusion (i).

To prove the necessity in (ii), consider any μ^2 in the set $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$ and any vector μ^1 in $\mathfrak{M}(\epsilon^1_1\phi\epsilon^2)$. Theorem (D) shows the existence of a vector ν^2 in $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2)$ such that

(a) $$\nu^1 = J^{2\epsilon_\kappa}\kappa^{12}\mu^2, \qquad \mu^2 = J^1\kappa^{*21}\nu^1.$$

By using the fact that κ^{12} is modular as to $\epsilon^1\epsilon^2_\kappa$, we have

(b) $$J^{2\epsilon_\kappa}(J^1\bar{\mu}^1\kappa^{12}, \mu^2) = J^1\bar{\mu}^1\nu^1.$$

Equations (a) and (b) show that the condition is necessary. The sufficiency can be proved similarly.

Now observe that κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_\kappa\epsilon^1_{\phi^*}\epsilon^2$ if and only if

(c) $$J^2(J^1\bar{\mu}^1\phi^{*12}, \eta^2) = J^1(\bar{\mu}^1, J^{2\epsilon_\kappa}\kappa^{12}\eta^2)$$

for every μ^1 in $\mathfrak{M}(\epsilon^1_{\phi^*}\phi\epsilon^2)$ and every η^2 in $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$. The latter condition is obviously satisfied if and only if (4.2) holds, since $\mathfrak{M}(\epsilon^1_{\phi^*}\phi\epsilon^2)$ and $\mathfrak{M}(\epsilon^2_{\phi^*}\cap\epsilon^2)$ are in one-to-one correspondence via the transformations $J^1\phi^{21}$ and $J^{2\phi^*}\phi^{*12}$ by (D).

THEOREM 5. *Suppose that* ϵ^1_0, ϵ^1_1, κ^{12}, ϕ^{21} *have the same properties as in the hypothesis of Theorem* 4, *and* $\mathfrak{M}(\epsilon^1_\kappa)$ *contains either* $\mathfrak{M}(\epsilon^1_1)$ *or* $\mathfrak{M}(\epsilon^1_0)$. *Then* κ^{12}, ϕ^{21} *are biorthogonal as to* $\epsilon^1\epsilon^1_0\epsilon^1_1\epsilon^2$ *if and only if* κ^{12}, ϕ^{21} *are biorthogonal as to* $\epsilon^1\epsilon^1_\kappa\epsilon^1_1\epsilon^2$.

PROOF. The condition is necessary by Theorem 4. If $\mathfrak{M}(\epsilon^1_\kappa)$ contains $\mathfrak{M}(\epsilon^1_0)$, the condition is obviously sufficient. If $\mathfrak{M}(\epsilon^1_\kappa)$ contains $\mathfrak{M}(\epsilon^1_1)$, the sufficiency is proved by the following argument: Let μ^1, ν^1 be vectors in $\mathfrak{M}(\epsilon^1_1\phi\epsilon^2)$, $\mathfrak{M}(\epsilon^1_0\kappa^*\epsilon^2)$ respectively. Since μ^1 is modular as to ϵ^1_κ by hypothesis, we have

$$J^{1\,1\,1}_{\bar{\mu}\,\nu} = J^1(J^{1\,1\,1}_{\bar{\mu}\,\epsilon_\kappa,\,\nu}) = J^1(\bar{\mu}^1, J^{1\,1\,1}_{\epsilon_\kappa\nu}).$$

Also

$$J^{1\,*21\,1}_{\kappa\quad\nu} = J^1J^{1\,*21\,1\,1}_{\kappa\quad\epsilon_\kappa\nu} = J^1(\kappa^{*21}, J^{1\,1\,1}_{\epsilon_\kappa\nu}).$$

The vector $J^1\epsilon^1_\kappa\nu^1$ is evidently in the set $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2)$.

COROLLARY 6. *Suppose that* κ^{12}, ϕ^{*12} *are by columns of* $\mathfrak{M}(\epsilon^1)$, $\mathfrak{M}(\epsilon^1_\kappa)$

respectively. Then κ^{12}, ϕ^{21} *are biorthogonal as to* $\epsilon^1\epsilon_0^1\epsilon_\kappa^1\epsilon^2$ *if and only if* κ^{12}, ϕ^{21} *are biorthogonal as to* $\epsilon^1\epsilon_\kappa^1\epsilon^2$.

THEOREM 7. *Suppose that* ϵ_0^1, ϵ_1^1, κ^{12}, ϕ^{21} *have the same properties as in the hypothesis of Theorem* 4, *and* $\mathfrak{M}(\epsilon_\kappa^2 \cap \epsilon^2)$ *is everywhere dense in* $\mathfrak{M}(\epsilon_\kappa^2)$. *Then* κ^{12}, ϕ^{21} *are biorthogonal as to* $\epsilon^1\epsilon_0^1\epsilon_1^1\epsilon^2$ *and the orthogonal complement within* $\mathfrak{M}(\epsilon_1^1)$ *of* $\mathfrak{M}(\epsilon_{\phi^*}^1)$ *is a subset of* $\mathfrak{M}(\epsilon_0^1)$ *if and only if* κ^{12}, ϕ^{21} *are biorthogonal as to* $\epsilon^1\epsilon_0^1\epsilon_{\phi^*}^1\epsilon^2$ *and* ϕ^{21} *is complete by rows-conjugate of* $\mathfrak{M}(\epsilon_1^1)$.

PROOF. The condition is obviously sufficient. For the necessity, we need to show only that ϕ^{21} is complete by rows-conjugate of $\mathfrak{M}(\epsilon_1^1)$. Consider any ξ^1 modular as to ϵ_1^1 such that $J^{11}\phi^{21}\xi^1 = J^1\phi^{21}\xi^1 = 0^2$. Then ξ^1 belongs to $\mathfrak{M}(\epsilon_1^1\phi\epsilon^2)$. Hence

$$J^1\bar{\mu}^1\xi^1 = J^2(J^1\bar{\mu}^1\kappa^{12}, J^1\phi^{21}\xi^1) = 0$$

for every μ^1 belonging to $\mathfrak{M}(\epsilon_0^1\kappa^*\epsilon^2)$. Now if $\mathfrak{M}(\epsilon_\kappa^2 \cap \epsilon^2)$ is everywhere dense in $\mathfrak{M}(\epsilon_\kappa^2)$, then $\mathfrak{M}(\epsilon_0^1\kappa^*\epsilon^2)$ is everywhere dense in $\mathfrak{M}(\epsilon_0^1)$, or, $O^1\mathfrak{M}(\epsilon_0^1\kappa^*\epsilon^2) = O^1\mathfrak{M}(\epsilon_0^1)$. Since ξ^1 is in $\mathfrak{M}(\epsilon_0^1)$ and also in $O^1\mathfrak{M}(\epsilon_0^1)$, it follows that $\xi^1 = 0^1$.

THEOREM 8. *Assume that* κ^{12} *is of type* $\mathfrak{M}(\epsilon^1)\overline{\mathfrak{M}}(\epsilon^2)$, *and* ϕ^{21} *is by rows-conjugate of* $\mathfrak{M}(\epsilon^1)$ *such that* κ^{12}, ϕ^{21} *are biorthogonal as to* $\epsilon^1\epsilon_\kappa^1\epsilon_{\phi^*}^1\epsilon^2$. *Then*

(1) ϕ^{*12}, κ^{*21} *are contraceding as to* $\epsilon^1\epsilon_{\phi^*}^2\epsilon^2$;

(2) $\mathfrak{M}(\epsilon^2 \cap \epsilon_{\phi^*}^2) \subset \mathfrak{M}(\epsilon^2\kappa\epsilon_\kappa^1)$;

(3) $\mu^1 = J^2\kappa^{12}J^1\phi^{21}\mu^1$ *for every* μ^1 *in* $\mathfrak{M}(\epsilon_{\phi^*}^1\phi\epsilon^2)$;

(4) $\mathfrak{M}(\epsilon^2 \cap \epsilon_{\phi^*}^2) \subset [J^1\phi^{21}\mu^1 \,|\, \mu^1$ *in* $\mathfrak{M}(\epsilon_\kappa^1 \cap \epsilon_{\kappa^*}^1)]$;

(5) $\mathfrak{M}(\epsilon_\kappa^1\phi\epsilon^2) \supset \mathfrak{M}(\epsilon_\kappa^1 \cap \epsilon_{\kappa^*}^1)$ *if and only if* $\mathfrak{M}(\epsilon^2\kappa\epsilon_\kappa^1)$ *contains* $[J^1\phi^{21}\mu^1 \,|\, \mu^1$ *in* $\mathfrak{M}(\epsilon_\kappa^1 \cap \epsilon_{\kappa^*}^1)]$.

PROOF. Since κ^{12} is by rows-conjugate of $\mathfrak{M}(\epsilon^2 \cap \epsilon_\kappa^2)$, we make use of (4.2) in Theorem 4 and secure

$$J^2\kappa^{12}\mu^2 = J^1\epsilon_\kappa^1 J^{2\phi^*}\phi^{*12}\mu^2 = J^{2\phi^*}\phi^{*12}\mu^2$$

for every μ^2 belonging to $\mathfrak{M}(\epsilon^2 \cap \epsilon_{\phi^*}^2)$. This proves (1). Part (2) is an immediate consequence of (1) just proved. For the demonstration of (3), we note that ϵ_κ^1 is by columns of $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ and hence by Definition 2, we have

$$J^{1\,1\,1}_{\epsilon_\kappa\eta} = J^2(J^{1\,1\,12}_{\epsilon_\kappa\kappa}, J^{1\,21\,1}_{\phi\,\eta}) = J^{2\,12}_{\kappa}J^{1\,21\,1}_{\phi\,\eta}$$

for every η^1 in $\mathfrak{M}(\epsilon_{\phi^*}^1\phi\epsilon^2)$. Since $\mathfrak{M}(\epsilon_{\phi^*}^1)$ is a subset of $\mathfrak{M}(\epsilon_\kappa^1)$, we have $J^1\epsilon_\kappa^1\eta^1 = \eta^1$, and hence, part (3). To prove (4), consider any μ^2 in

$\mathfrak{M}(\epsilon^2 \cap \epsilon^2_{\phi^*})$. Let μ^1 be a vector in $\mathfrak{M}(\epsilon^1_\kappa \cap \epsilon^1_{\kappa^*})$ for which $J^{2\phi^*}\phi^{*12}\mu^2 = \mu^1$. Then by (1)

$$J^1\phi^{21}\mu^1 = J^1\phi^{21}J^{2\phi^*}\phi^{*12}\mu^2 = J^{2\phi^*}{}_{\epsilon^1_{\phi^*}}\mu^2 = \mu^2,$$

whence (4) follows. In part (5), if $\mathfrak{M}(\epsilon^1_\kappa\phi\epsilon^2) \supset \mathfrak{M}(\epsilon^1_\kappa \cap \epsilon^1_{\kappa^*})$, then

$$[J^1\phi^{21}\mu^1 \mid \mu^1 \text{ in } \mathfrak{M}(\epsilon^1_\kappa\phi\epsilon^2)] \supset [J^1\phi^{21}\mu^1 \mid \mu^1 \text{ in } \mathfrak{M}(\epsilon^1_\kappa \cap \epsilon^1_{\kappa^*})].$$

By (C), the left-hand side is identical with $\mathfrak{M}(\epsilon^2 \cap \epsilon^2_{\phi^*})$, which by part (2) is a subset of $\mathfrak{M}(\epsilon^2\kappa\epsilon^1_\kappa)$. The converse is obvious.

THEOREM 9. *Let ϵ^1_0 be idempotent as to ϵ^1, and κ^{12} be of type $\mathfrak{M}(\epsilon^1_0)\overline{\mathfrak{M}}(\epsilon^2)$. Suppose that ϕ^{21} is by rows-conjugate of $\mathfrak{M}(\epsilon^1_0)$ and κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^2$. Then we have the following conclusions*:

(1) *ϕ^{21} is complete by rows-conjugate of $\mathfrak{M}(\epsilon^1_0)$*;

(2) *$\mu^1 = J^2\kappa^{12}J^1\phi^{21}\mu^1$ for every μ^1 in $\mathfrak{M}(\epsilon^1_0\phi\epsilon^2)$*;

(3) *$\mathfrak{M}(\epsilon^1_0\phi\epsilon^2) = [J^2\kappa^{12}\mu^2 \mid \mu^2$ in $\mathfrak{M}(\epsilon^2 \cap \epsilon^2_{\phi^*})] \subset \mathfrak{M}(\epsilon^1_0 \cap \epsilon^1_{\kappa^*})$*;

(4) *$\mathfrak{M}(\epsilon^2 \cap \epsilon^2_{\phi^*}) \subset \mathfrak{M}(\epsilon^2\kappa\epsilon^1_0)$*;

(5) *if ϕ^{21} is by columns of $\mathfrak{M}(\epsilon^2)$, then $\epsilon^1_0 = J^2\kappa^{12}\phi^{21}$ and κ^{12} is complete by columns of $\mathfrak{M}(\epsilon^1_0)$*;

(6) *$\mathfrak{M}(\epsilon^1_0\phi\epsilon^2) \supset \mathfrak{M}(\epsilon^1_0 \cap \epsilon^1_{\kappa^*})$ if and only if every vector in $\mathfrak{M}(\epsilon^1_0 \cap \epsilon^1_{\kappa^*})$ is expressible in the form $J^2\kappa^{12}\mu^2$, where μ^2 is a vector in $\mathfrak{M}(\epsilon^2 \cap \epsilon^2_{\phi^*})$.*

PROOF. Part (1) follows from Theorem 7, for $\mathfrak{M}(\epsilon^2 \cap \epsilon^2_\kappa)$ is everywhere dense in $\mathfrak{M}(\epsilon^2_\kappa)$ when κ^{12} is of type $\mathfrak{M}(\epsilon^1_0)\overline{\mathfrak{M}}(\epsilon^2)$. Part (2) is proved in the same way as part (3) of Theorem 8 with the replacement of ϵ^1_κ by ϵ^1_0. By Theorem (C), part (3) is a consequence of (2). Part (4) follows from (3). For part (5), we note that ϵ^1_0 is by columns of $\mathfrak{M}(\epsilon^1_0\phi\epsilon^2)$; hence the first conclusion follows from (2) whereas the second follows from (1) and the fact that ϕ^{*12}, κ^{*21} are biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^2$. Part (6) is a consequence of (3).

THEOREM 10. *Suppose that κ^{12} is of type $\mathfrak{M}(\epsilon^1)\overline{\mathfrak{M}}(\epsilon^2)$, ϕ^{21} is by rows-conjugate of $\mathfrak{M}(\epsilon^1_\kappa)$, and κ^{12}, ϕ^{21} are biorthogonal as to $\epsilon^1\epsilon^1_\kappa\epsilon^2$. Then the following assertions are equivalent*:

(1) *$\mathfrak{M}(\epsilon^2 \cap \epsilon^2_{\phi^*}) \supset \mathfrak{M}(\epsilon^2\kappa\epsilon^1_\kappa)$*;

(2) *κ^{12} is complete by rows-conjugate of $\mathfrak{M}(\epsilon^2)$ and $\mathfrak{M}(\epsilon^1_\kappa\phi\epsilon^2)$ contains $\mathfrak{M}(\epsilon^1_\kappa \cap \epsilon^1_{\kappa^*})$*;

(3) *κ^{12} is complete by rows-conjugate of $\mathfrak{M}(\epsilon^2)$ and $\mathfrak{M}(\epsilon^2\kappa\epsilon^1_\kappa)$ contains $[J^1\phi^{21}\mu^1 \mid \mu^1$ in $\mathfrak{M}(\epsilon^1_\kappa \cap \epsilon^1_{\kappa^*})]$*;

(4) *κ^{*21}, ϕ^{*12} are contraceding as to $\epsilon^2\epsilon^1_{\kappa^*}\epsilon^1$, and κ^{12} is complete by rows-conjugate of $\mathfrak{M}(\epsilon^2)$*;

(5) $[J^1\phi^{21}\mu^1 \mid \mu^1$ *in* $\mathfrak{M}(\epsilon_\kappa^1 \cap \epsilon_{\kappa^*}^1)] \supset \mathfrak{M}(\epsilon^2 \kappa \epsilon_\kappa^1)$.

If one of the preceding five conditions is valid, then $\epsilon^2 = \epsilon_{\kappa^*}^2 = J^1\phi^{21}\kappa^{12}$.

PROOF. The equivalence of the second and third conditions follows from (5) of Theorem 8. Conditions (1), (2), (4), and (5) are equivalent because of Theorem 2, where κ, ϕ, ϵ^1, ϵ^2, ϵ_ϕ^1 are replaced by κ^*, ϕ^*, ϵ^2, ϵ_κ^1, $\epsilon_{\phi^*}^2$ respectively. The relation $\epsilon^2 = J^1\phi^{21}\kappa^{12}$ follows from (4), since κ^{12} is by columns of $\mathfrak{M}(\epsilon_\kappa^1 \cap \epsilon_{\kappa^*}^1)$.

THE UNIVERSITY OF CHICAGO

ON SPHERICAL CYCLES[1]

SAMUEL EILENBERG

Given a metric separable space Y, we consider the homology group $B^n(Y)$ obtained using n-dimensional singular cycles in Y with integer coefficients. Every continuous mapping $f \in Y^{S^n}$ of the oriented n-dimensional sphere S^n into Y defines uniquely an element $h(f)$ of $B^n(Y)$. Clearly if $f_0, f_1 \in Y^{S^n}$ are two homotopic mappings, then $h(f_0) = h(f_1)$.

The homology classes $h(f)$ will be called *spherical homology classes*. A cycle will be called *spherical* if its homology class is spherical.[2]

THEOREM 1. *If Y is arcwise connected, the spherical homology classes form a subgroup of $B^n(Y)$.*

Let $p \in S^n$, $q \in Y$, and let $S^n = S^n_+ + S^n_-$ be a decomposition of S^n into two hemispheres such that $p \in S^n_+ \cdot S^n_-$. Consider $f_0, f_1 \in Y^{S^n}$. It is well known that, replacing if necessary f_0 and f_1 by homotopic mappings, we may assume that $f_0(S^n_+) = q$ and that $f_1(S^n_-) = q$. Defining $f = f_0$ on S^n_- and $f = f_1$ on S^n_+ we clearly have

$$f \in Y^{S^n}, \qquad h(f) = h(f_0) + h(f_1).$$

The homology class $h(f_0) + h(f_1)$ is therefore spherical.

Let M^r be an r-dimensional (finite or infinite) manifold[3] and P^{r-n-1} $(n > 0)$ an at most $(r-n-1)$-dimensional subpolyhedron of M^r.

THEOREM 2. *Every n-dimensional cycle γ^n in $M^r - P^{r-n-1}$ such that $\gamma^n \sim 0$ in M^r is spherical (with respect to $M^r - P^{r-n-1}$).*

Let a^{r-n-1} be an $(r-n-1)$-dimensional simplex of M^r and b^{n+1} the $(n+1)$-cell dual to it. The boundary ∂b^{n+1} is contained in $M^r - P^{r-n-1}$ and is a spherical cycle. Since $M^r - P^{r-n-1}$ is connected, the spherical homology classes of $B^n(M^r - P^{r-n-1})$ form a group. It follows that each cycle of the form

$$(*) \qquad \partial\left(\sum_i \alpha_i b_i^{n+1}\right)$$

is a spherical cycle with respect to $M^r - P^{r-n-1}$. The cycle γ^n is homologous in $M^r - P^{r-n-1}$ to a cycle of the form (*). Therefore γ^n is spherical.

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[1] Presented to the Society, April 13, 1940.

[2] Spherical cycles were considered by W. Hurewicz, Proceedings, Akademie van Wetenschappen te Amsterdam, vol. 38 (1935), pp. 521–528.

[3] See K. Reidemeister, *Topologie der Polyeder*, Leipzig, 1938, p. 151.

THEOREM 3. *Let γ^n be a spherical cycle in M^r and let $r>2n$. Then there is a simplicial homeomorphism*[4] *$g\in M^{S^n}$ such that $\gamma^n\in h(g)$.*

This is an immediate consequence of Theorem 5 below. Using Theorem 2 we obtain the following:

THEOREM 4. *Given an n-cycle $\gamma^n\subset M^r-P^{r-n-1}$ $(r>2n)$ such that $\gamma^n\sim 0$ in M^r, there is a cycle $\gamma_1^n\subset M^r-P^{r-n-1}$ which is a simplicial and homeomorphic image of S^n such that $\gamma^n\sim\gamma_1^n$ in M^r-P^{r-n-1}.*

THEOREM 5. *Let Q^n be a finite n-dimensional polyhedron and let $r>2n$. Every continuous mapping $f\in M^{Q^n}$ can be approached by simplicial homeomorphisms $g\in M^{Q^n}$.*

We may admit that the mapping f is simplicial. Let $a_1, a_2, \cdots, a_k$ be the vertices of the complex $f(Q^n)$ and let $\sigma_1, \sigma_2, \cdots, \sigma_k$ be the corresponding stars.[5] Let us choose $\delta>0$ so that $x\in f(Q^n)$ will imply $\rho(x, M^r-\sigma_i)>\delta$ for some $i=1, 2, \cdots, k$.

Let $\delta>2\epsilon>0$. We are going to define a sequence $f=f_0, f_1, \cdots, f_k$ of simplicial maps of Q^n into M^r such that

$$|f_i(x)-f_{i-1}(x)|<\frac{\epsilon}{k}, \tag{1}$$

$$f_i(x_1)=f_i(x_2) \quad \text{implies} \quad f_{i-1}(x_1)=f_{i-1}(x_2), \tag{2}$$

$$x_1\neq x_2 \text{ and } f_i(x_1)=f_i(x_2)=y \text{ imply } \rho(y, M^r-\sigma_i)<\delta\frac{2k-i}{2k}. \tag{3}$$

Suppose that $f_0, f_1, \cdots, f_{i-1}$ are already defined. Let

$$f_i(x)=f_{i-1}(x) \quad \text{if} \quad f_{i-1}(x)\in M^n-\sigma_i,$$

and let $Q_i^k=f_{i-1}^{-1}(\sigma_i)$.

M^r being a manifold, σ_i is simplicially homeomorphic with a convex r-cell in a euclidean r-dimensional space. Since $r>2n$, then using the very well known[6] procedure of making vertices linearly independent we find a simplicial map $f_i(Q_i^n)\subset\sigma_i$ such that $f_i(x)=f_{i-1}(x)$ if $f_{i-1}(x)$ is on the boundary of σ_i and satisfying (1)–(3).

Taking $g=f_k$ it follows from (1) that

$$|g(x)-f(x)|<\epsilon.$$

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[4] With respect to certain simplicial subdivisions of M^r and S^n.

[5] σ_i consists of all closed simplices of M^r containing a_i.

[6] See for instance W. Hurewicz, Sitzungsberichte der Preussischen Akademie der Wissenschaften, vol. 24 (1933), p. 758.

Now if $x_1 \neq x_2$ and $g(x_1) = g(x_2)$, then according to (2) we have

$$f_i(x_1) = f_i(x_2) = y_i \qquad \text{for } i = 0, 1, \cdots, k.$$

Owing to the definition of δ there is an index $j = 0, 1, \cdots, k$ such that

$$\rho(y_0, M^n - \sigma_j) > \delta.$$

Combining this with (1) we see that

$$\rho(y_i, M^n - \sigma_j) > \delta - \frac{\epsilon i}{k} > \delta - \frac{\delta i}{2k} = \delta \frac{2k - i}{2k}.$$

Taking $i = j$ we obtain a contradiction with (3).

UNIVERSITY OF MICHIGAN

A NEW CONCEPT IN DISTANCE GEOMETRY WITH APPLICATIONS TO SPHERICAL SUBSETS[1]

LEONARD M. BLUMENTHAL

1. **Introduction.** A semimetric space is a set of abstract elements (*points*) with a structure imposed by attaching to each pair p, q of elements a non-negative real number pq (*distance*) such that $pq=qp$ and $pq=0$ if and only if $p=q$. Some of the more important of these spaces (for example, euclidean, hyperbolic, and spherical spaces of n-dimensions) have the exceedingly useful property that the existence of a function mapping an arbitrary semimetric space congruently (that is, *with preservation of distances*) upon the space follows from the congruent embedding in the space of each set of k points (not necessarily pairwise distinct) of the semimetric space.[2] Such spaces are said to have *congruence order* k with respect to all semimetric spaces—a term introduced by Menger, who proved that the n-dimensional euclidean space E_n has this property with $n+3$ as the smallest value of k.[3]

The importance of the possession by a given space of a congruence order is clear, since it evidently reduces the characterization problem of determining necessary and sufficient conditions for the congruent embedding of *any* semimetric space in the given one to the finite problem of finding under what conditions a semimetric set of k points is congruently contained in the space. The desirability of having k as small as possible is obvious.

After showing that $n+3$ is the minimum congruence order of the n-dimensional euclidean space E_n, Menger proved the more difficult theorem that if a semimetric space S is not congruently contained in the E_n, but each set of $n+2$ of its points is congruent with $n+2$ points of the E_n, then S consists of exactly $n+3$ points. Hence the congruent embedding in the E_n of each set of $n+2$ points of a semimetric space S insures that S is congruently contained in E_n *whenever* S *has more than* $n+3$ *points*. This property was labelled "quasi congruence order

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[1] Presented to the Society, April 12, 1940.

[2] See Chapter III of the author's *Distance Geometries*, University of Missouri Studies, vol. 13, 1938.

[3] The proof is given in Menger, *Untersuchungen über allgemeine Metrik*, Mathematische Annalen, vol. 100 (1928), pp. 75–163. The term "congruence order $n+3$" describing this property was introduced in *Bemerkungen zur zweiten Untersuchung über allgemeine Metrik*, Proceedings of the Royal Academy, Amsterdam, vol. 30 (1927), pp. 1–5.

$n+2$."[4] It is enjoyed by the n-dimensional hyperbolic space also, but not by the n-dimensional elliptic nor the n-dimensional spherical spaces. The notions of congruence and quasi congruence orders have been the principal ones employed in the metric characterizations of spaces.

Recent investigations of characterization problems for spherical *subsets* (that is, subsets of the surface of the sphere with geodesic (shorter arc) metric) have revealed the inadequacy of these two notions to describe the results obtained.[5] It is the purpose of this paper to present a general concept which systematizes all of these results, contains the older notions as two links of a chain of notions which occur in a special case, and which orders between these two links infinitely many other links. This new concept gives rise to a whole field of new problems in distance geometry and, besides, induces a re-examination of older, solved problems with a view towards a "better" solution. The meaning of the latter part of the above remark will become clear later.

2. **σ-relative congruence indices.** Let S and σ be two given semimetric spaces. Then S has *σ-relative congruence indices (n, k) with respect to a given class (Σ) of semimetric spaces* provided any space Σ of (Σ) with more than $n+k$ pairwise distinct points is congruently contained in S whenever each n of its points (not necessarily pairwise distinct) is congruently contained in σ. It seems appropriate to call σ a catalytic space of index (n, k) of S with respect to the class (Σ), since it facilitates the desired reaction of congruent embedding of Σ in S but plays in general no part in the result itself, for Σ is not necessarily embeddable in σ. If $\sigma = S$ the indices (n, k) are called *congruence indices* of S with respect to (Σ). It is this case with which the present paper is principally concerned.

REMARK 1. *A semimetric space S has congruence indices $(n, 0)$ with respect to (Σ) if and only if S has congruence order n with respect to (Σ).*

For suppose S has congruence indices $(n, 0)$ with respect to (Σ) and let Σ be any space of (Σ) with each n of its points congruently contained in S. If Σ contains more than n pairwise distinct points, then Σ is congruently contained in S by hypothesis and definition. On the other hand, if Σ consists of n or fewer pairwise distinct points, the

[4] The proof appears in *Untersuchungen über allgemeine Metrik*, loc. cit., but the term "quasi congruence order" is first defined in Menger, *New foundations of euclidean geometry*, American Journal of Mathematics, vol. 53 (1931), pp. 721–745.

[5] C. V. Robinson, *Contributions to Distance Geometry*, University of Missouri thesis.

congruence of Σ with a subset of S is a trivial consequence of the congruent embedding in S of each n points of Σ. Hence S has congruence order n with respect to (Σ).

Finally, if S has congruence order n with respect to (Σ), then congruence of any space Σ of (Σ) follows from the congruent embedding in S of each n points of Σ, and the situation described by S having congruence indices $(n, 0)$ with respect to (Σ) evidently exists.

REMARK 2. *A semimetric space S has congruence indices $(n, 1)$ with respect to (Σ) if and only if S has the quasi congruence order n with respect to (Σ).*

The proof is similar to that of Remark 1.

Thus the notions of congruence and quasi congruence orders correspond to particular σ-relative congruence indices in the special case $\sigma = S$.

Let $(n, k) \rightarrow (n', k')$ symbolize the statement, "if S has congruence indices (n, k) with respect to (Σ), then S has congruence indices (n', k') with respect to (Σ)."

REMARK 3. *If $n \leqq n'$ and $n+k \leqq n'+k'$, then*

$$(n, k) \rightarrow (n', k').$$

The proof follows immediately from the definition of congruence indices.

Some special cases of Remark 3 are of interest. Thus, $(n, 1) \rightarrow (n+1, 0)$; that is, quasi congruence order n implies congruence order $n+1$. Also $(n, 1) \rightarrow (n, 2) \rightarrow (n+1, 1)$. This relation, which interpolates a property between quasi congruence order n and quasi congruence order $n+1$ illustrates one of the advantages of the concept of congruence indices. Previous to its introduction, if a space S were found not to have quasi congruence order n, the next step was to examine it for a higher quasi congruence order. Consideration was thus shifted from the congruent embedding in S of sets of n points to the more difficult problems concerning embedding in S of sets of $n+1$ or more points. The possession by S of congruence indices $(n, 2)$ implies quasi congruence order $n+1$, but, in addition, characterizes S in terms of the congruent embedding in S of each n points of a semimetric space rather than each $n+1$ points.

In the light of the foregoing observations, any one of the congruence symbols (n, k), $(k = 0, 1, 2, \cdots)$, is to be preferred to the symbol $(n+1, 0)$, though only for $k = 0, 1$ is the implication $(n, k) \rightarrow (n+1, 0)$ valid. Similarly, the symbols $(n, 1)$, $(n, 2)$, $(n, 3)$, $\cdots$ are each preferable to $(n+1, 1)$, though only the first two imply the latter. Thus,

there is interpolated between quasi congruence orders n and $n+1$ an infinity of symbols, each of which is to be preferred to quasi congruence order $n+1$. It is clear that the indices (n, k) characterize a space only with respect to all semimetric spaces *containing more than* $n+k$ *points*, but this slight restriction of the class of comparison spaces is more than offset by the simplicity gained from keeping the number of points to be embedded as small as possible. In addition to the possibility of bettering old results (e.g., the possession of indices $(n, 2)$ where previously only $(n+1, 1)$ and not $(n, 1)$ had been proved), new problems arise (e.g., the possibility of indices $(n, 3)$ when neither $(n, 1)$ nor $(n+1, 1)$ exists for the space).

The above considerations suggest the following preferential (lexicographical) ordering of congruence indices. If (n, k) and (n', k') are two congruence symbols of S, that one is preferred for which (1) the first index is the smaller or (2) if $n=n'$, then the preferred symbol has the smaller second index. That symbol (n, k) of S is *best* provided no symbol of S is preferred to it. It is emphasized that this ordering is *not* that of logical implication. With respect to the latter criterion congruence symbols form only a partially ordered set (for example, the symbols $(n, 3)$ and $(n+1, 1)$ are not logically comparable).

Since $(n, k) \rightarrow (n+k, 0)$, S has no congruence indices with respect to (Σ) if S has no congruence order with respect to (Σ). Conversely, if S has congruence order m with respect to (Σ), then by Remark 1, S has congruence indices with respect to (Σ). Thus, S possesses congruence indices if and only if S has a congruence order.

3. **Lemmas concerning pseudo-spherical sets.** The n-dimensional spherical space $S_{n,r}$ of radius r is the "surface" of a sphere of radius r in a euclidean space of $n+1$ dimensions, with geodesic (shorter arc) metric. The $S_{n,r}$ has indices $(n+3, 0)$ but not $(n+2, 1)$, and there exist semimetric spaces of arbitrary power exceeding $n+2$ which are not contained congruently in $S_{n,r}$ though every set of $n+2$ points is congruently embeddable in $S_{n,r}$. Such spaces are called pseudo-$S_{n,r}$ sets. From the characterization of these spaces (given elsewhere) we take the following results:[6]

PROPERTY 1. *Let P be a pseudo-$S_{n,r}$ set of more than $n+3$ pairwise distinct points, no two of which have distance πr. If p, q are any two distinct points of P, then*

$$\cos(pq/r) = \pm 1/(n+1).$$

[6] L. M. Blumenthal and G. R. Thurman, *The characterization of pseudo-spherical sets*, American Journal of Mathematics, vol. 62 (1940), pp. 835–854.

PROPERTY 2. *Let P be a pseudo-$S_{n,r}$ set of m pairwise distinct points*

$$p_1, p_2, \cdots, p_m, \qquad m > n + 3$$

no two of which have distance πr, and consider the determinant

$$\Delta_m(p_1, p_2, \cdots, p_m) = \left| \cos (p_i p_j / r) \right|, \qquad i, j = 1, 2, \cdots, m,$$

of the set. Apart from the elements of the first and jth rows, $(j = 2, 3, \cdots, m)$, either each element in the first column of Δ_m is equal to the corresponding element in the jth column, or each element in the first column is the negative of the corresponding element in the jth column. If the first column of Δ_m contains at least one positive element different from the element 1, then apart from the elements in the first and second rows, each element in the second column is the negative of the corresponding element in the first column. Due to the symmetry of Δ_m and the first part of this property, the sign of every element of Δ_m is then determined.[7]

PROPERTY 3. *If P is a pseudo-$S_{n,r}$ set of more than $n+3$ pairwise distinct points, no two of which have distance πr, then every set of m points of P, $(m \geqq n+3)$, is a pseudo-$S_{n,r}$ set.*

LEMMA 1. *Apart from the labelling, there is a unique pseudo-$S_{n,r}$ set of $2n+2$ pairwise distinct points, $(n>1)$, free from diametral point-pairs (that is, pairs of points with distance πr), which does not contain an equilateral set of $n+2$ points (that is, a set of $n+2$ points with all distances equal).*[8]

PROOF. Let P be any pseudo-$S_{n,r}$ set of $2n+2$ points, no two of which have distance πr. Since $n>1$, $2n+2>n+3$, and the determinant Δ_{2n+2} of P has Properties 1, 2 listed above. To prove the lemma it suffices to show that apart from a (symmetric) shifting of rows and columns, there is a single distribution of the signs in Δ_{2n+2} which will not yield a principal minor of order $n+2$ with all elements outside the principal diagonal negative. (Since each set of $n+2$ points of P is congruently contained in $S_{n,r}$, an $(n+2)$-nd order principal minor of Δ_{2n+2} with all positive elements is impossible. Thus an equilateral $(n+2)$-tuple has each distance equal to $r \cdot \cos^{-1} -1/(n+1)$.)

Applying Property 2, a brief consideration of Δ_{2n+2} shows that the first column of the determinant must contain *exactly* $n+1$ positive elements (apart from 1) and an appropriate (symmetric) shifting of

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[7] See L. M. Blumenthal, *Metric methods in determinant theory*, American Journal of Mathematics, vol. 61 (1939), pp. 912–922.

[8] This lemma is not valid for $n=1$.

rows and columns gives (schematically):

$$\begin{array}{r}\left.\begin{array}{c}\\\\\\\\\\\end{array}\right\}\\ n+2\\ \left.\begin{array}{c}\\\\\\\\\\\end{array}\right\}\\ n\\ \left.\begin{array}{c}\\\\\\\\\\\end{array}\right\}\end{array}\left|\begin{array}{ccccccccc} 1 & + & + & \cdot & + & - & - & \cdot & - \\ + & 1 & - & \cdot & - & + & + & \cdot & + \\ + & - & 1 & \cdot & - & + & + & \cdot & + \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ + & - & - & \cdot & 1 & + & + & \cdot & + \\ - & + & + & \cdot & + & 1 & - & \cdot & - \\ - & + & + & \cdot & + & - & 1 & \cdot & - \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ - & + & + & \cdot & + & - & - & \cdot & 1 \end{array}\right|$$

COROLLARY. *Every pseudo-$S_{n,r}$ set of $2n+2$ pairwise distinct points, $(n>1)$, no two with distance πr, contains an equilateral $(n+1)$-tuple with each distance equal to $r\cdot\cos^{-1}-1/(n+1)$.*

For the only such set without an equilateral $(n+2)$-tuple is, apart from labelling, the set exhibited in the determinant of Lemma 1. Deleting the 2nd, 3rd, $\cdots$, $(n+2)$-nd rows and columns of this determinant, one obtains the determinant of a set of the kind specified in the corollary. Another such set is given by deleting the 1st, $(n+3)$-rd, $\cdots$, $(2n+2)$-nd rows and columns.

LEMMA 2. *Each pseudo-$S_{n,r}$ set P of more than $2n+2$ pairwise distinct points, $(n>1)$, no two of which have distance πr, contains an equilateral $(n+2)$-tuple.*[9]

PROOF. Since $n>1$, $2n+2>n+3$, and hence P contains more than $n+3$ pairwise distinct points. By Property 3, each set $p_1, p_2, \cdots, p_{2n+3}$ of $2n+3$ points of P is a pseudo-$S_{n,r}$ set. It suffices to show that these points contain an equilateral $(n+2)$-tuple. Again by Property 3, the points $p_1, p_2, \cdots, p_{2n+2}$ form a pseudo-$S_{n,r}$ set of more than $n+3$ points, and according to Lemma 1 either these points contain an equilateral $(n+2)$-tuple or they constitute a set of the kind exhibited in the proof of that lemma.

In the latter case, consider the determinant Δ_{2n+3} of the points $p_1, p_2, \cdots, p_{2n+3}$. The principal minor of order $2n+2$ in the upper left-hand corner of Δ_{2n+3} is the determinant of Lemma 1. If the element in the last row and first column of Δ_{2n+3} is positive, then

[9] We suppose $n>1$ since Lemma 1 is applied. It is easily shown that Lemma 2 is valid for $n=1$.

the principal minor of order $n+2$ obtained by deleting the 1st, $(n+4)$-th, $\cdots$, $(2n+3)$-rd rows and columns of Δ_{2n+3} has all its elements outside the principal diagonal negative, and hence the corresponding $(n+2)$-tuple is equilateral. If, on the other hand, the element in the last row and first column of Δ_{2n+3} is negative, deleting the 2nd, 3rd, $\cdots$, $(n+2)$-nd rows and columns yields the determinant of an equilateral $(n+2)$-tuple.

4. **Applications to spherical subsets.** The subsets of the $S_{n,r}$ to which the foregoing is applied in this paper are spherical caps $\kappa_{n,\rho}$ of spherical radius ρ, $0<\rho\leqq\pi r/2$ (that is, the locus of points of $S_{n,r}$ whose distance from a fixed point of $S_{n,r}$ is less than or equal to ρ).

THEOREM 1. *Let σ denote the spherical cap (n-dimensional hemisphere) $\kappa_{n,\pi r/2}$, $n>1$, with base circle removed. Then $S_{n,r}$ has σ-relative congruence indices $(n+2, n)$ with respect to all semimetric spaces* (Σ).[10]

PROOF. We must show that each semimetric space Σ containing more than $2n+2$ pairwise distinct points and having each $n+2$ of them congruent with $n+2$ points of σ is congruently contained in $S_{n,r}$. Suppose this is not the case. Then Σ is a pseudo-$S_{n,r}$ set of more than $2n+2$ points, no two of which have distance πr (since each $n+2$ points of Σ are congruently contained in σ) and hence, by Lemma 2, Σ contains an equilateral set of $n+2$ points with each of the $\frac{1}{2}(n+1)(n+2)$ mutual distances equal to $r\cdot\cos^{-1}-1/(n+1)$. But this is not possible, for such an $(n+2)$-tuple is clearly not contained in σ.[11] This contradiction yields the theorem. Obvious examples show that Σ is not necessarily contained in σ.

Thus the open n-dimensional hemisphere is a catalytic set of index $(n+2, n)$ of the containing sphere.

THEOREM 2. *Let $\kappa_{n,\rho}$, $n>1$, be an n-dimensional spherical cap of radius ρ, $(\rho<\pi r/2)$. Then $\kappa_{n,\rho}$ has congruence indices $(n+2, n)$ with respect to all semimetric spaces* (Σ).

PROOF. Let Σ be any semimetric space containing more than $2n+2$ points, each $n+2$ of which are congruently contained in $\kappa_{n,\rho}$. Then it

[10] Since any arc $\kappa_{1,\rho}$, $\rho\leqq\pi r/2$, of a circle $S_{1,r}$ is evidently congruent with a straight line segment (which has indices (3, 1) with respect to (Σ)), Theorems 1, 2 of this section hold for $n=1$.

[11] In the contrary case there exist $n+2$ positive numbers $b_1, b_2, \cdots, b_{n+2}$ such that the $(n+3)$rd order determinant obtained by bordering the determinant of the equilateral set by these numbers (with 1 in the intersection of the bordering row and column) is zero. This yields quite easily $\sum_{i=1}^{n+2} b_i=0$, which is impossible.

follows from Theorem 1 that Σ is congruently contained in $S_{n,r}$. But it has been shown that *with respect to subsets of* $S_{n,r}$, the cap $\kappa_{n,\rho}$, $(\rho<\pi r/2)$, has congruence indices $(n+2, 0)$.[12] Hence Σ is congruently contained in $\kappa_{n,\rho}$, and the theorem is proved.

REMARK 1. Since $(n+2, n)\rightarrow(2n+1, 1)$, the cap $\kappa_{n,\rho}$, $(\rho<\pi r/2)$, has quasi congruence order $2n+1$ with respect to semimetric spaces. As n increases, however, the indices on the left-hand side of the above implication tend to place conditions on only *half* the number of points that must be isometrically embedded if the characterization is given in terms of the indices on the right-hand side of the implication.

THEOREM 3. *The cap* $\kappa_{n,\rho}$, $(\rho<r\cdot\cos^{-1}1/(n+1))$, $n>1$, *has congruence indices* $(n+2, n-1)$ *with respect to all semimetric spaces* (Σ).[13]

PROOF. If Σ is any semimetric space of more than $2n+1$ pairwise distinct points, each $n+2$ of which are embeddable in $\kappa_{n,\rho}$, $(\rho<r\cdot\cos^{-1}1/(n+1))$, then Σ is congruently contained in $S_{n,r}$, for an assumption to the contrary implies that Σ is a pseudo-$S_{n,r}$ set of at least $2n+2$ pairwise distinct points (no two with a distance πr). Then by the corollary to Lemma 1, Σ contains an equilateral $(n+1)$-tuple with each distance equal to $r\cdot\cos^{-1}-1/(n+1)$. Such an $(n+1)$-tuple is not embeddable in any cap of radius $\rho<r\cdot\cos^{-1}1/(n+1)$. Since, now, Σ is congruently contained in $S_{n,r}$ and each cap $\kappa_{n,\rho}$ of radius $\rho<\pi r/2$ has congruence indices $(n+2, 0)$ with respect to subsets of $S_{n,r}$, it follows that Σ is congruently contained in $\kappa_{n,\rho}$, $(\rho<r\cdot\cos^{-1}1/(n+1))$.

REMARK 1. For $n=2$, the cap $\kappa_{2,\rho}$, $(\rho<r\cdot\cos^{-1}1/3)$, has indices $(4,1)$; that is, quasi congruence order 4 with respect to all semimetric spaces. Thus spherical caps with radii less than $r\cdot\cos^{-1}1/3$ behave like the *euclidean plane* in this important respect. The indices $(4,1)$ cannot be improved no matter how small ρ is taken.

REMARK 2. For $n=3$, the indices $(5,2)$ of the cap $\kappa_{3,\rho}$, $\rho<r\cdot\cos^{-1}1/4$, given by Theorem 3, can be bettered. A brief consideration shows that every pseudo-$S_{3,r}$ set of *more than six points* (no two with distance πr) contains at least one equilateral set of four points with mutual distances equal to $r\cdot\cos^{-1}-1/4$. A proof quite similar to that of Theorem 3 demonstrates that $\kappa_{3,\rho}$, $\rho<r\cdot\cos^{-1}1/4$, has the indices $(5,1)$ and hence behaves like the euclidean three-dimensional spaces in this respect.

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[12] C. V. Robinson, loc. cit.

[13] This theorem is not valid for $n=1$.

REMARK 3. If $n=4$, the indices (6,3) given by Theorem 3 for the cap $\kappa_{4,\rho}$, $(\rho < r\cdot\cos^{-1} 1/5)$, cannot be reduced to (6,1) by a simple argument analogous to that presented in the preceding remark.[14] This arises from the fact that there exists a pseudo-$S_{4,r}$ set of eight points (no two diametral) that does not contain an equilateral quintuple with mutual distances equal to $r\cdot\cos^{-1}-1/5$. Such a pseudo-$S_{4,r}$ set is represented schematically by

$$\begin{vmatrix} 1 & + & + & + & + & - & - & - \\ + & 1 & - & - & - & + & + & + \\ + & - & 1 & - & - & + & + & + \\ + & - & - & 1 & - & + & + & + \\ + & - & - & - & 1 & + & + & + \\ - & + & + & + & + & 1 & - & - \\ - & + & + & + & + & - & 1 & - \\ - & + & + & + & + & - & - & 1 \end{vmatrix}$$

There is no fifth-order principal minor of this determinant with all elements outside the principal diagonal negative.

The problems of (1) finding the best indices for caps $\kappa_{n,\rho}$, $\rho < r\cdot\cos^{-1} 1/(n+1)$, $(n>1)$, and (2) ascertaining for each n the n-dimensional cap of maximum spherical radius ρ that, *like the n-dimensional euclidean space*, has congruence indices $(n+2, 1)$, will be investigated in a later paper.

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[14] It is easily shown that the indices (6,3) can be reduced to (6,2). Whether the cap $\kappa_{4,\rho}$, $(\rho < r\cos^{-1}\frac{1}{5})$, does behave like the four-dimensional euclidean space with regard to the symbol (6,1) is not known.

THE FINITE DIFFERENCES OF POLYGENIC FUNCTIONS[1]

RUFUS P. ISAACS

By a polygenic function $f(z)$ we shall mean a function analytic in x and y separately, but whose real and imaginary parts are not required to satisfy the Cauchy-Riemann equations. At any point z the derivative of such a function will depend on θ, the angle at which the incremented point (used in defining the derivative) approaches z. The set of these numbers, for a fixed z, but for different θ, form a circle. The equation for the derivative was given by Riemann in his classic dissertation (1851), but Kasner was the first to point out that it was a circle and make a detailed study of its geometry.[2] Hedrick called it the Kasner circle.

In this paper we shall be concerned with the finite difference quotients of polygenic functions. We shall show how a surface can be constructed for each point z representing the difference quotient, and the derivative circle is a cross section of this surface.

The conjugate form. Regard

$$z = x + iy, \qquad \bar{z} = x - iy$$

as a linear substitution, and perform its inverse

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z - \bar{z})$$

on $f(z)$. The resulting $F(z, \bar{z})$ will be called the conjugate form of f. Let $D_z F$ and $D_{\bar{z}} F$ be the partial derivatives[3] of $F(z, \bar{z})$, regarding z and $\bar{z}$ as independent variables. That is,

$$D_z F = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \tfrac{1}{2}(D_x - D_y)f,$$

(1)

$$D_{\bar{z}} F = \tfrac{1}{2}(D_x + D_y)f.$$

The operator E^ω. Let $\omega = \rho e^{i\theta}$. We define

$$E^\omega f(z) = f(z + \omega).$$

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[1] Presented to the Society, February 25, 1939, under the title *A geometric interpretation of the difference quotient of polygenic functions.*

[2] *General theory of polygenic or non-monogenic functions; The derivative congruence of circles*, Proceedings of the National Academy of Sciences, vol. 13 (1928), pp. 75–82. *A new theory of polygenic functions*, Science, vol. 66 (1927). Also, *The Geometry of Polygenic Functions*, Kasner and DeCicco—a book in the course of preparation.

[3] In Kasner's notation, these are $\mathfrak{M}(f)$ and $\mathfrak{P}(f)$.

More precisely, this means

$$E^{\omega}f(x, y) = f(x + \rho \cos \theta, y + \rho \sin \theta).$$

E may also have a "partial" meaning:

$$E_x^a f(x, y) = f(x + a, y).$$

And we see the equivalence of the two operators:

$$E^{\omega} = E_x^{\rho \cos \theta} \cdot E_y^{\rho \sin \theta}. \tag{2}$$

Taylor's expansion may be written in the form

$$E_x^a f(x, y) = \left(1 + aD_x + \frac{1}{2!} a^2 D_x^2 + \cdots\right) f(x, y) = \exp(aD_x) f. \tag{3}$$

Now, by combining (2) and (3) we obtain the operational equivalence

$$E^{\omega} = \exp(\rho \cos \theta \, D_x + \rho \sin \theta \, D_y).$$

$(\cos \theta \, D_x + \sin \theta \, D_y) f$ is nothing more than the directional derivative of f, which we designate by $\mathcal{D}f$. Substituting from (1) for D_x and D_y, we see that

$$\mathcal{D} = e^{i\theta} D_z + e^{-i\theta} D_{\bar{z}}, \tag{4}$$

$$E^{\omega} = \exp(\rho \mathcal{D}). \tag{5}$$

The differential quotient. We define

$$_{\omega}\Delta f(z) = \frac{f(z + \omega) - f(z)}{\omega},$$

whence, operationally,

$$_{\omega}\Delta = \frac{1}{\omega} [E^{\omega} - 1].$$

Expanding by means of (4) and (5)

$$_{\omega}\Delta f(z) = \frac{1}{\rho e^{i\theta}} \left[\rho \mathcal{D} + \frac{\rho^2 \mathcal{D}^2}{2!} + \cdots\right] f(z)$$

$$= \left[D_z + e^{-2i\theta} D_{\bar{z}} + \frac{\rho}{2!} (e^{i\theta} D_z^2 + 2e^{-i\theta} D_z D_{\bar{z}} + e^{-3i\theta} D_{\bar{z}}^2) + \cdots\right] f(z). \tag{6}$$

To obtain the derivative we let $\rho \to 0$. The resulting expression

$$[D_z + e^{-2i\theta} D_{\bar{z}}] f(z),$$

which Kasner calls γ, is immediately seen to be the points of a circle, when z is fixed and θ varies.

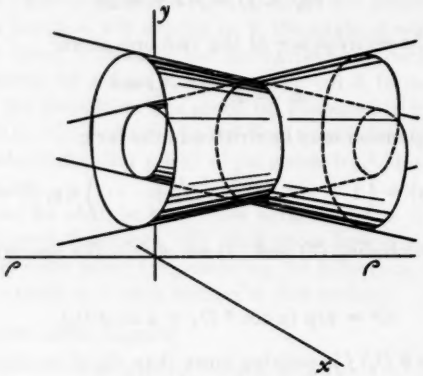

FIG. 1

For a geometric interpretation of (6), let us add a third coordinate ρ to the x, y plane. In this 3-space, (6) represents a surface for each fixed value of z. Such a surface we will call a Kasnercoid or K-coid of the function.

Example. $f(z) = x^2 + y^2 + i(x+y)$. Its conjugate form is $z\bar{z} + \frac{1}{2}(i+1)z + \frac{1}{2}(i-1)\bar{z}$. Using (6) we obtain for the difference quotient:

$$_{\rho}\Delta f = \bar{z} + \tfrac{1}{2}(i+1) + e^{-2i\theta}[z + \tfrac{1}{2}(i-1)] + \rho e^{-i\theta}.$$

The K-coid of a typical point is sketched in Figure 1.

This of course is a very simple case. In regard to the nature of K-coids in general, we state a theorem.

A curve will be called a doubler of a circle when it has the following properties:

1. In traversing the doubler once, we pass around the center of the circle twice.
2. No circle intersects the doubler in more than six points.
3. Every point on the circle is the midpoint of two points on the doubler.

(See Figure 2.)

THEOREM. *If ρ^2 be neglected, then sections of the K-coid sufficiently near the derivative circle are doublers of the derivative circle.*

Compare with the monogenic case:

For a monogenic function, the derivative circle reduces to a point and the nearby sections are nearly circles.

The purpose of this paper is to throw some light on the nature of derivatives. We may consider the K-coid compressed along the ρ-axis

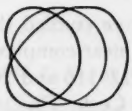

FIG. 2

so as to be a two-sheeted (at least for small ρ) Riemann surface with the derivative circle acting as "branch circle." Let us consider the familiar expression:

$$\lim_{\omega \to 0} \frac{f(z+\omega) - f(z)}{\omega},$$

and suppose $z+\omega$, in approaching z, travels along some curve. To each point on the curve correspond two points on the K-coid which approach a common point on the derivative circle.

Connection with the second derivative. Let us write (6) in the form

$$[e^{-i\theta}D + (\omega/2!)e^{-2i\theta}D^2 + \cdots]f(z).$$

Is the coefficient of $\omega/2!$ the second derivative of $f(z)$? This depends on what is meant by the second derivative, as there are several alternative methods of defining it.[4] According to Kasner's method,[5] the second derivative depends not only on the slope of the path of the incremented point but also on the curvature of the path. The expression is

$$[e^{-2i\theta}D^2 - 2ie^{-3i\theta}\kappa D_s]f(z),$$

where κ is the curvature.

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[4] See my unpublished Master's Essay, Columbia University.

[5] *The second derivative of a polygenic function*, Transactions of this Society, vol. 30, no. 4.

The corresponding expression for what I call the type A derivative—based on another, but equally logical definition—is merely the first term of the above expression.

COLUMBIA UNIVERSITY

ON THE ASYMPTOTIC LINES OF A RULED SURFACE

GUIDO FUBINI

Many mathematicians have studied the surfaces every *asymptotic curve* of which belongs to a linear complex. I will here be content with the results given on pages 112–116 and 266–288 of a treatise[1] written by myself and Professor A. Cech. This treatise gives (p. 113) a very simple proof of the following theorem:

If every non-rectilinear asymptotic curve of a ruled surface S belongs to a linear complex, all these asymptotic curves are projective to each other.

We will find all the ruled surfaces, the non-rectilinear asymptotic curves of which *are projective to each other*, and prove conversely that *every one of these asymptotic curves belongs to a linear complex*. If c, c' are two of these asymptotic curves and if A is an arbitrary point of c, we can find on c' a point A' such that the straight line AA' is a straight generatrix of S. The projectivity, which, according to our hypothesis, transforms c into c', will carry A into a point A_1 of c'. We will prove that *the two points A' and A_1 are identical*; but this theorem is not obvious and therefore our demonstration cannot be very simple. The generalization to nonruled surfaces seems to be rather complicated: and we do not occupy ourselves here with such a generalization.

If the point $x=x(u, v)$ generates a ruled surface S, for which $u=$const. and $v=$const. are asymptotic curves, we can suppose (loc. cit., p. 182)

$$x = y + uz \tag{1}$$

in which y and z are functions of v. More clearly, if x_1, x_2, x_3, x_4 are homogeneous projective coordinates of a point of S, we can find eight functions y_i and z_i of v such that

$$x_i = y_i(v) + uz_i(v), \qquad i = 1, 2, 3, 4. \tag{1$_{bis}$}$$

From the general theory of surfaces, it is known (loc. cit., p. 90) that

[1] *Geometria Proiettiva Differenziale*, Bologna, Zanichelli.

we can find five functions θ, β, γ, p_{11}, p_{22} of u, v such that

$$(2)\qquad \begin{cases} x_{uu} = \theta_u x_u + \beta x_v + p_{11}x, \\ x_{vv} = \gamma x_u + \theta_v x_v + p_{22}x, \end{cases}$$

$$x_u = \frac{\partial x}{\partial u},\quad \theta_u = \frac{\partial \theta}{\partial u},\quad x_{uu} = \frac{\partial^2 x}{\partial u^2},\quad \cdots;$$

$$x = x_i;\quad i = 1, 2, 3, 4;\quad x = y + uz.$$

Since now $x_{uu}=0$, the former of these equations becomes

$$0 = \theta_u x_u + \beta x_v + p_{11}x,$$

and therefore (since the points x, x_u, x_v are independents):

$$\theta_u = \beta = p_{11} = 0.$$

Equation (2) becomes

$$y_{vv} + uz_{vv} = \gamma z + \theta_v(y_v + uz_v) + p_{22}(y + uz).$$

And, by differentiating two times with respect to u,

$$0 = \frac{\partial^2 p_{22}}{\partial u^2}\, y + \frac{\partial^2(\gamma + up_{22})}{\partial u^2}\, z.$$

Therefore

$$0 = \frac{\partial^2 p_{22}}{\partial u^2} = \frac{\partial^2}{\partial u^2}(\gamma + up_{22})$$

and we can write

$$(3)\qquad p_{22} = A + Bu, \qquad \gamma + up_{22} = C + Du,$$

$$(4)\qquad \gamma = (C + Du) - \mu(A + Bu),$$

in which A, B, C, D are functions only of v. We can multiply the x_i or, what is the same, the y_i and the z_i by a factor of proportionality (function *only of* v) such that $\theta=$const., and $\theta_v=0$ (or that the determinant of the y_i, z_i, $y_i' = \partial y_i/\partial v$, $z_i' = \partial z_i/\partial v$ becomes a constant). The second equation of (2) becomes

$$x_{vv} = \gamma z + px, \qquad p = p_{22} = A + Bu,$$

and, by differentiating with respect to u,

$$z_{vv} = Dz + By = (D - uB)z + Bx.$$

Therefore

$$\gamma z = x_{vv} - px, \qquad p = A + Bu = p_{22},$$

$$\frac{\partial^2}{\partial v^2}\frac{x_{vv} - px}{\gamma} - (D - uB)\frac{x_{vv} - px}{\gamma} - Bx = 0;$$

or

$$x'''' - 2\frac{\gamma'}{\gamma}x''' + \left[2\left(\frac{\gamma'}{\gamma}\right)^2 - \frac{\gamma''}{\gamma} - A - D\right]x'' + 2\frac{pq' - p'q}{\gamma}x'$$

$$+\left[2\frac{\gamma'}{\gamma}\frac{p'q - pq'}{\gamma} + \frac{pq'' - p''q}{\gamma} + AD - BC\right]x = 0, \tag{5}$$

$$p = A + Bu, \qquad q = C + Du, \qquad \gamma = q - up,$$

$$p' = \frac{\partial p}{\partial v}, \qquad \gamma' = \frac{\partial \gamma}{\partial v}, \qquad x' = \frac{\partial x}{\partial v}, \cdots.$$

This is the differential equation which defines the asymptotic curves $u = \text{const.}$ If we put $x = X\gamma^{1/2}$, this equation becomes

$$X'''' + lX'' + mX' + nX = 0$$

in which

$$l = 2\frac{\gamma''}{\gamma} - \frac{5}{2}\left(\frac{\gamma'}{\gamma}\right)^2 - (A + D), \qquad n = -\frac{35}{16}\left(\frac{\gamma'}{\gamma}\right)^4 + \frac{r}{\gamma^3},$$

$$m = 2\frac{\gamma'''}{\gamma} - 7\frac{\gamma'\gamma''}{\gamma^2} + 5\left(\frac{\gamma'}{\gamma}\right)^3 + 2\frac{pq' - qp'}{\gamma} - (A + D)\frac{\gamma'}{\gamma}$$

(r is a polynomial of the variable u). The projective invariants (or covariants) of the curve defined by this equation are

$$Udv^3, \qquad V_1dv^2, \qquad Wdv^4.$$

We have put

$$U = l' - m = \frac{\epsilon}{\gamma}, \qquad [\epsilon = \{(A' - D')C - (A - D)C'\}$$

$$+ 2(CB' - BC')u + \{(A' - D')B - B'(A - D)\}u^2],$$

$$W = 20l'' - 50m' - 9l^2 + 100n = k\left(\frac{\gamma'}{\gamma}\right)^4 + \frac{R}{\gamma^3},$$

$$k = \text{const.} = 175 \neq 0; \ R \text{ a polynomial of } u,$$

and (if $U \neq 0$)

$$V_1 = 6[\log U]'' - \left(\frac{U'}{U}\right)^2 - \frac{36}{5}l, \qquad U' = \frac{\partial U}{\partial v}.$$

If $U=0$, the curve belongs to a linear complex; if $U\neq 0$,

$$\frac{W^3}{U^4}=\frac{(k\gamma'^4+R\gamma)^3}{\gamma^8\epsilon^4}$$

is a projective invariant. If all the asymptotic curves $u=$const. are projective to each other, this ratio must not be dependent upon u. And therefore the values of u, for which $\gamma=0$, must also satisfy the equation $\gamma'=0$. Therefore

$$\frac{C'}{D'}=\frac{D'-A'}{D-A}=\frac{B'}{B}$$

or

$$\gamma=C+(D-A)u-Bu^2=V(c+bu+au^2)\cdot$$

(V function of v; a, b, $c=$const.) Therefore $\epsilon=0$, $U=0$ and every asymptotic curve $u=$const. belongs to a linear complex. In this case

$$\frac{\gamma'}{\gamma}=\frac{V'}{V},\qquad \frac{\gamma''}{\gamma}=\frac{V''}{V},\qquad B'=\frac{V'}{V}B,$$

$$\frac{pq'-p'q}{\gamma}=\frac{p(q'-up')-p'(q-up)}{\gamma}=p\frac{\gamma'}{\gamma}-p'$$

$$=(A+Bu)\frac{V'}{V}-\left(A'+Bu\frac{V'}{V}\right)=\frac{AV'-A'V}{V}\cdot$$

And analogously

$$\frac{pq''-p''q}{\gamma}=\frac{AV''-A''V}{V}\cdot$$

Therefore no one of the coefficients of (5) is dependent upon u, and consequently we can suppose that the projectivity, which carries an asymptotic curve $u=$const. into another, carries every point of the former into that point of the latter which belongs to the same rectilinear generatrix of the surface (because the corresponding value of v is not changed by this projectivity).

We have in this manner completely demonstrated the stated theorems.

THE INSTITUTE FOR ADVANCED STUDY

A SEQUENCE OF LIMIT TESTS FOR THE CONVERGENCE OF SERIES[1]

MARGARET MARTIN

In this paper, we shall develop a sequence of limit tests for the convergence and divergence of infinite series of positive terms which is similar in form to the De Morgan and Bertrand sequence but involves the ratio of two successive values of the test ratio rather than the test ratio itself. The proof will be based on the following integral test by R. W. Brink:[2]

"THEOREM VI. *Given the sequence* $\{u_n\}$. *Let* $r_n = u_{n+1}/u_n$ *and* $R_n = r_{n+1}/r_n = u_{n+2}u_n/u_{n+1}^2$. *If* $\lim_{n=\infty} r_n = 1$, *and if* $R(x)$ *is a function such that* $R(n) = R_n$, *and such that* $R(x) \geqq R(x')$ *when* $x' > x$, *a necessary and sufficient condition for the convergence of the series* $\sum_{n=0}^{\infty} u_n$ *is the convergence of the integral*

$$\int_0^{\infty} \exp\left\{-\int_0^x \int_x^{\infty} \log R(x)\, dx dx\right\} dx."$$

Since a finite number of terms does not affect convergence or divergence, the conditions of Theorem VI need hold only for n greater than some fixed number ν, in which case zero is to be replaced by ν as a lower limit of integration.

The foregoing theorem admits a generalization similar to that given by C. T. Rajagopal[3] in the case of another theorem of Brink's.[4] However, Brink's Theorem VI is sufficient for the purposes of the present paper.

LEMMA. *Let* $\{u_n\}$ *and* $\{u_n'\}$ *be sequences of positive terms with ratios* $r_n = u_{n+1}/u_n$, $R_n = r_{n+1}/r_n$, $r_n' = u_{n+1}'/u_n'$, *and* $R_n' = r_{n+1}'/r_n'$, *such that* $\lim_{n\to\infty} r_n = \lim_{n\to\infty} r_n' = 1$.

1. *If the series* $\sum_{n=\nu}^{\infty} u_n'$ *converges and if* $R_n \geqq R_n'$ *for all values of* $n \geqq \nu$, *then the series* $\sum_{n=\nu}^{\infty} u_n$ *converges.*

2. *If the series* $\sum_{n=\nu}^{\infty} u_n'$ *diverges and if* $R_n \leqq R_n'$ *for all values of* $n \geqq \nu$, *then the series* $\sum_{n=\nu}^{\infty} u_n$ *diverges.*

[1] Presented to the Society, April 27, 1940, under the title *A sequence of tests for the convergence and divergence of infinite series.*

[2] R. W. Brink, *A new sequence of integral tests for the convergence and divergence of infinite series*, Annals of Mathematics, vol. 21 (1919), pp. 39–60.

[3] C. T. Rajagopal, *On an integral test of R. W. Brink for the convergence of series*, this Bulletin, vol. 43 (1937), pp. 405–412.

[4] R. W. Brink, *A new integral test for the convergence and divergence of infinite series*, Transactions of this Society, vol. 19 (1918), pp. 186–204.

PROOF. In case 1, if $n \geqq \nu$,

$$\frac{r_{n+1}}{r_n} \geqq \frac{r'_{n+1}}{r'_n}, \quad \frac{r_{n+2}}{r_{n+1}} \geqq \frac{r'_{n+2}}{r'_{n+1}}, \cdots, \quad \frac{r_{N+1}}{r_N} \geqq \frac{r'_{N+1}}{r'_N}.$$

Multiplying these inequalities, we have $r_{N+1}/r_n \geqq r'_{N+1}/r'_n$, and taking the limit as N becomes infinite, we obtain $r_n \leqq r'_n$, $(n \geqq \nu)$. Hence if $\sum_{n=\nu}^{\infty} u'_n$ converges, $\sum_{n=\nu}^{\infty} u_n$ also converges. A similar proof can be given for the case of divergence.

THEOREM 1. *Let* $\{u_n\}$ *be a sequence of positive terms with ratios* $r_n = u_{n+1}/u_n$, $R_n = r_{n+1}/r_n$, *such that* $\lim_{n\to\infty} r_n = 1$. *If of the limits*

$$\lim_{n\to\infty} n^2 \log R_n = a_0,$$

$$\lim_{n\to\infty} \log n(n^2 \log R_n - 1) = a_1,$$

$$\lim_{n\to\infty} \log \log n\{\log n(n^2 \log R_n - 1) - 1\} = a_2,$$

$$\lim_{n\to\infty} \log \log \log n[\log \log n\{\log n(n^2 \log R_n - 1) - 1\} - 1] = a_3,$$

. . . . . .

a_k is the first which is finite and different from 1, *or the first to be positively or negatively infinite, the series* $\sum_{n=\nu}^{\infty} u_n$ *converges if* $a_k > 1$ *and diverges if* $a_k < 1$.

PROOF. Let $l_1 = \log n$, $l_k = \log l_{k-1}$, $(k > 1)$;

$$L_k(n, \alpha) = \frac{1}{l_1} + \frac{1}{l_1 l_2} + \cdots + \frac{1}{l_1 l_2 \cdots l_{k-1}} + \frac{\alpha}{l_1 l_2 \cdots l_k}, \quad k \geqq 1,$$

$$L_0(n, \alpha) = 0.$$

By hypothesis,

$$(1) \quad \lim_{n\to\infty} l_k[l_{k-1}\{l_{k-2}(\cdots l_2[l_1(n^2 \log R_n - 1) - 1]\cdots) - 1\} - 1] = a_k.$$

Hence

$$(2) \quad \lim_{n\to\infty} \frac{1}{1+L_k(n,1)}[l_k\{l_{k-1}(l_{k-2}[\cdots l_2\{l_1(n^2 \log R_n - 1) - 1\}\cdots] - 1) - 1\} - l_k l_{k-1}\cdots l_2 L_1(n,1) - l_k l_{k-1}\cdots l_3 L_2(n,1) - \cdots - l_k l_{k-1} L_{k-2}(n,1) - l_k L_{k-1}(n,1)] = a_k,$$

since $\lim_{n\to\infty} 1/[1+L_k(n,1)] = 1$ and $\lim_{n\to\infty} l_k l_{k-1}\cdots l_j L_{j-1}(n,1) = 0$, $(1 \leqq j \leqq k)$.

(a) If $a_k>1$, let α_1 be a number such that $1<\alpha_1<a_k$. Let N_1 be chosen sufficiently large so that for $n\geqq N_1$, $l_k(n)$ is defined and positive and

$$\begin{aligned}(3)\quad &\frac{1}{1+L_k(n,1)}\left[l_k\left\{l_{k-1}(l_{k-2}[\cdots l_2\left\{l_1(n^2\log R_n-1)-1\right\}\cdots]-1)-1\right\}\right.\\ &-l_kl_{k-1}\cdots l_2L_1(n,1)-l_kl_{k-1}\cdots l_3L_2(n,1)-\cdots-l_kl_{k-1}L_{k-2}(n,1)\\ &\left.-l_kL_{k-1}(n,1)\right]>\alpha_1.\end{aligned}$$

It follows that

$$\begin{aligned}(4)\quad \log R_n>{}&\frac{1}{n^2}+\frac{1}{n^2l_1}\left[1+L_1(n,1)\right]+\frac{1}{n^2l_1l_2}\left[1+L_2(n,1)\right]+\cdots\\ &+\frac{1}{n^2l_1l_2\cdots l_{k-1}}\left[1+L_{k-1}(n,1)\right]\\ &+\frac{\alpha_1}{n^2l_1l_2\cdots l_k}\left[1+L_k(n,1)\right].\end{aligned}$$

Let

$$\begin{aligned}M_k(x,\alpha)={}&\frac{1}{x^2}+\frac{1+l_1}{x^2l_1^2}+\frac{1+l_2+l_2l_1}{x^2l_1^2l_2^2}+\cdots\\ &+\frac{1+l_{k-1}+l_{k-1}l_{k-2}+\cdots+l_{k-1}\cdots l_1}{x^2l_1^2l_2^2\cdots l_{k-1}^2}\\ &+\alpha\frac{1+l_k+l_kl_{k-1}+\cdots+l_k\cdots l_1}{x^2l_1^2l_2^2\cdots l_k^2},\end{aligned}$$

where now $l_1=\log x$, and so on. Then (4) can be written

$$(5)\qquad \log R_n>M_k(n,\alpha_1).$$

(b) If $a_k<1$, let α_2 be a positive number such that $a_k<\alpha_2\leqq 1$. Proceeding as in (a), we can show that for n greater than a suitably chosen number N_2,

$$(6)\qquad \log R_n<M_k(n,\alpha_2).$$

Consider the series

$$(7)\qquad \sum_{n=\nu}^{\infty}u'_{n;\alpha}\qquad u'_{n;\alpha}=\exp\left\{-\sum_{j=\nu}^{n-1}\sum_{i=j}^{\infty}M_k(i,\alpha)\right\},\qquad \nu>N_1,N_2.$$

For this series, $r'_n=\exp\left\{-\sum_{i=n}^{\infty}M_k(i,\alpha)\right\}$, $R'_n=\exp\left\{M_k(n,\alpha)\right\}$. It is easily shown that the conditions of Brink's Theorem VI are satis-

fied with $R'(x)=\exp\{M_k(x,\alpha)\}$ and $n\geqq\nu$. The test integral then has the form

$$\int_\nu^\infty \exp\left\{-\int_\nu^x\int_x^\infty M_k(x,\alpha)dxdx\right\}dx = K\int_\nu^\infty \frac{1}{xl_1l_2\cdots l_{k-1}l_k^\alpha}dx,$$

where K is constant. This integral, and hence the series $\sum_{n=\nu}^\infty u'_{n;\alpha}$, converges for $\alpha>1$ and diverges for $\alpha\leqq 1$. We now apply the lemma to the series $\sum_{n=\nu}^\infty u_n$ and $\sum_{n=\nu}^\infty u'_{n;\alpha}$.

In case (a), we set $\alpha=\alpha_1$ in series (7). $\sum_{n=\nu}^\infty u'_{n;\alpha_1}$ converges since $\alpha_1>1$. From (5), we have $R_n>\exp\{M_k(n,\alpha_1)\}=R_n$, $n\geqq\nu$. The conditions of part 1 of the lemma are satisfied, and hence $\sum_{n=\nu}^\infty u_n$ converges.

In case (b), we set $\alpha=\alpha_2$ in series (7). $\sum_{n=\nu}^\infty u'_{n;\alpha_2}$ diverges, and from (6),

$$R_n < \exp\{M_k(n,\alpha_2)\} = R'_n, \qquad n\geqq\nu.$$

Hence $\sum_{n=\nu}^\infty u_n$ diverges by part 2 of the lemma.

The tests of Theorem 1 apply to series for which an explicit expression for R_n is known. The general term of such a series has the form $u_n=\prod_{j=\nu}^{n-1}\prod_{m=j}^\infty\phi(m)$.

Example 1. Consider the series $\sum_{n=3}^\infty u_n$, where

$$u_n = \exp\left\{-\sum_{j=2}^{n-1}\sum_{m=j}^\infty \phi(m)\right\}, \qquad \phi(m)=\frac{\alpha+\beta\log(m^2)}{m^2\log(m^2)}, \qquad \beta>0.$$

We have $r_n=\exp\{-\sum_{m=n}^\infty\phi(m)\}$; $\lim_{n\to\infty} r_n=1$ since $\sum_{m=3}^\infty\phi(m)$ converges for all values of α and β; $R_n=\exp\{\phi(n)\}$; $\log R_n=\phi(n)$. We apply the first test of Theorem 1,

$$\lim_{n\to\infty} n^2\log R_n = \lim_{n\to\infty}\frac{\alpha+\beta\log(n^2)}{\log(n^2)} = \beta.$$

Thus the series converges for $\beta>1$ and diverges for $\beta<1$, regardless of the value of α. For the case $\beta=1$, we apply the second test,

$$\lim_{n\to\infty}\log n(n^2\log R_n-1) = \lim_{n\to\infty}\log n\left[\frac{\alpha+\log(n^2)}{\log(n^2)}-1\right] = \frac{\alpha}{2},$$

and the series converges for $\beta=1$, $\alpha>2$, and diverges for $\beta=1$, $\alpha<2$. For the case $\beta=1$, $\alpha=2$, we go on to the third limit test,

$$\lim_{n\to\infty}\log\log n[\log n(n^2\log R_n-1)-1]=0,$$

and the series diverges.

The tests of Theorem 1 are valid if $\log R_n$ is replaced by $R_n - 1$, the tests of the resulting sequence being in some cases more convenient to apply than those of Theorem 1.

THEOREM 2. *Let* $\{u_n\}$ *be a sequence of positive terms with ratios* $r_n = u_{n+1}/u_n$, $R_n = r_{n+1}/r_n$, *such that* $\lim_{n\to\infty} r_n = 1$. *If of the limits*

$$\lim_{n\to\infty} n^2(R_n - 1) = b_0,$$

$$\lim_{n\to\infty} \log n\left[n^2(R_n - 1) - 1\right] = b_1,$$

$$\lim_{n\to\infty} \log\log n\left\{\log n\left[n^2(R_n - 1) - 1\right] - 1\right\} = b_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

b_k *is the first which is finite and different from* 1, *or the first to be positively or negatively infinite, the series* $\sum_{n=\nu} u_n$ *converges if* $b_k > 1$ *and diverges if* $b_k < 1$.

PROOF. Proceeding as in the proof of Theorem 1, we are led to the following inequalities:

(a) If $b_k > 1$, then for any number β_1 such that $1 < \beta_1 < b_k$, and for n greater than a suitably chosen number N_1',

$$R_n - 1 > M_k(n, \beta_1). \tag{8}$$

(b) If $b_k < 1$, then for any positive number β_2 such that $b_k < \beta_2 \leqq 1$, and for n greater than a suitably chosen number N_2',

$$R_n - 1 < M_k(n, \beta_2). \tag{9}$$

In case (a), consider the series $\sum_{n=\nu} u_n''$,

$$u_n'' = \exp\left\{-\sum_{j=\nu}^{n-1}\sum_{i=j}^{\infty}\left[M_k(i, \beta_1) - (M_k(i, \beta_1))^2\right]\right\}, \qquad \nu \geqq N_1'.$$

It can be shown that this series satisfies the conditions of Brink's Theorem VI with $R''(x) = \exp\left\{M_k(x, \beta_1) - (M_k(x, \beta_1))^2\right\}$. The test integral has the form

$$\int_\nu^\infty \exp\left\{-\int_\nu^x \int_x^\infty \left[M_k(x, \beta_1) - (M_k(x, \beta_1))^2\right] dx\,dx\right\} dx$$

$$\leqq \int_\nu^\infty \exp\left\{-\int_\nu^x \int_x^\infty M_k(x, \beta_1)\,dx\,dx + \int_\nu^\infty \int_x^\infty (M_k(x, \beta_1))^2\,dx\,dx\right\} dx$$

$$= K' \int_\nu^\infty \exp\left\{-\int_\nu^x \int_x^\infty M_k(x, \beta_1)\,dx\,dx\right\} dx,$$

where K' is a constant. Since $\beta_1 > 1$, the latter integral converges, and hence the series $\sum_{n=\nu}^{\infty} u_n''$ converges. From (8),

$$R_n > 1 + M_k(n, \beta_1), \qquad n \geqq \nu.$$

Hence if ν is sufficiently large so that $M_k(\nu, \beta_1) < 1$,

$$\log R_n > \log [1 + M_k(n, \beta_1)] > M_k(n, \beta_1) - (M_k(n, \beta_1))^2 = \log R_n'', \qquad n \geqq \nu,$$

$R_n > R_n''$, and the series $\sum_{n=\nu}^{\infty} u_n$ of our theorem converges by part 1 of the lemma.

In case (b), set $\alpha = \beta_2$ in series (7), and take $\nu = N_2'$ and sufficiently large so that $|R_n' - 1| < 1$ for $n \geqq \nu$. $\sum_{n=\nu}^{\infty} u_{n;\beta_2}'$ diverges since $\beta_2 \leqq 1$. From (9), we have

$$R_n - 1 < M_k(n, \beta_2) = \log R_n' < R_n' - 1, \qquad n \geqq \nu,$$

$R_n < R_n'$, and hence the series $\sum_{n=\nu}^{\infty} u_n$ diverges by part 2 of the lemma.

Example 2. Consider the series $\sum_{n=3}^{\infty} u_n$, $u_n = \prod_{k=2}^{n-1} \prod_{m=k}^{\infty} (1 - \alpha/m^{\beta})$, $(\alpha > 0, \beta > 1)$. Here

$$r_n = \sum_{m=n}^{\infty} \left(1 - \frac{\alpha}{m^{\beta}}\right), \qquad R_n = \frac{1}{1 - \alpha/n^{\beta}}, \qquad R_n - 1 = \frac{\alpha}{n^{\beta} - \alpha}.$$

$\operatorname{Lim}_{n \to \infty} r_n = 1$, since $\prod_{m=3}^{\infty} (1 - \alpha/m^{\beta})$ converges. Applying the first test of Theorem 2, we find

$$\lim_{n \to \infty} n^2 (R_n - 1) = \lim_{n \to \infty} n^2 \cdot \frac{\alpha}{n^{\beta} - \alpha} = \begin{cases} +\infty, & \beta < 2, \\ 0, & \beta > 2, \\ \alpha, & \beta = 2. \end{cases}$$

Thus the series converges when $1 < \beta < 2$ and diverges when $\beta > 2$. If $\beta = 2$, the series converges for $\alpha > 1$ and diverges for $\alpha < 1$. If $\beta = 2$ and $\alpha = 1$, we apply the second test of the sequence,

$$\lim_{n \to \infty} \log n [n^2 (R_n - 1) - 1] = \lim_{n \to \infty} \log n \left[n^2 \frac{1}{n^2 - 1} - 1 \right] = 0,$$

and hence the series diverges.

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THE DOUBLE COSETS OF A FINITE GROUP[1]

J. S. FRAME

1. **Introduction.** It is the purpose of this paper to study some of the properties of the double cosets of a finite group and to prove two main theorems which generalize the results of two previous papers by the author,[2] giving some relations between the double cosets and the irreducible components of the permutation group generated by a given subgroup. We let H be an arbitrary but fixed subgroup of order h of a finite group G of order g, $g=nh$, and we let G_H be the permutation group of degree n induced by right multiplication of the cosets HS_i, $i=1, 2, \cdots, n$, by elements of G. When written as a group of permutation matrices and completely reduced, the group G_H will have r' distinct irreducible components Γ_i of degree n_i and multiplicity μ_i^H, and we may write

$$G_H = \sum_{i=1}^{r'} \mu_i^H \Gamma_i. \tag{1.1}$$

Multiplication of a right coset HG_k on the left by a single element of G does not in general produce a right coset, but if each coset HG_k is multiplied on the left by all the elements of a right coset HS_t and the products are added, a transformation is obtained which carries each of the n right cosets HG_k into a collection of right cosets $\sum_{\delta=1}^{k_t} HS_tH_\delta G_k$ in which, as we shall see in §4, each of the k_t cosets occurs $d_t=h/k_t$ times. Its matrix d_tV_t is permutable with each of the matrices of G_H. Certain cosets, which we shall call associated cosets, are permuted among themselves when multiplied on the right by elements of H. Each of these produces the same matrix V_t. The totality K_t of elements belonging to a complete set of k_t associated cosets, each counted once, will be called a *double coset*, whereas the term *weighted double coset* will refer to the complex of h^2 elements HS_tH in which each element of the double coset K_t occurs $d_t=n/k_t$ times. The integer d_t will be called the density. The number of distinct double cosets K_t will be denoted by r, and the elements S_t, $t=1, 2, \cdots, r$, one from each, will be said to *generate* the double

[1] Presented to the Society, September 12, 1940.

[2] J. S. Frame, *The degrees of the irreducible components of simply transitive permutation groups*, Duke Mathematical Journal, vol. 3 (1937), pp. 8–17.

J. S. Frame, *On the decomposition of transitive permutation groups generated by the symmetric group*, Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 132–139.

cosets. The double coset which contains the inverses of the elements of K_t will be denoted by $K_{t'}$.

In §2 a number of elementary properties of double cosets are stated, mostly without proof. In §3 a study is made of self-inverse double cosets, $K_t = K_{t'}$, and their number is shown to equal $(1/g)\sum\chi(R^2)$, where $\chi(R^2)$ is the trace of R^2 in the permutation group G_H, and where the sum is taken for all R in G. Using the results of a paper of Frobenius[3] we then prove the first principal theorem, which generalizes a result obtained by the author for the symmetric group.[4]

THEOREM A. *The number of self-inverse double cosets of a finite group G with respect to a subgroup H is equal to the sum of the multiplicities of those irreducible components of G_H which have a symmetric bilinear invariant minus the sum of the multiplicities of those which have an alternating bilinear invariant.*

The Hermitian invariants of G_H associated with the matrices V_t are studied in §4.[5] Two bases consisting of r independent invariants are found, the one obtained directly from the r double cosets, and the other from the $r = \sum_{t=1}^{r'}(\mu_t^H)^2$ Hermitian invariants[6] which come into evidence when G_H is completely reduced. The complex multiplication of the double cosets plays an important role in the discussion which culminates in the proof of Theorem B, which includes as a special case a theorem conjectured but only partially proved in a previous paper.[7]

THEOREM B. *Given a transitive permutation group G_H of degree n in which the subgroup H leaving one symbol fixed permutes the n symbols in r transitive sets of k_t symbols, $t = 1, 2, \cdots, r$. Let K be the product $\prod_{t=1}^{r'}(k_t)$ and let N, $N = \prod_{i=1}^{r'} n_i^{(\mu_i^H)^2}$, denote the product of the degrees n_i of the distinct irreducible components Γ_i of G_H each raised to a power equal to the square of its multiplicity in G_H. Then $n^{r-2}K/N = \overline{P}_1 P_1$, where P_1 is an algebraic integer in the field of the characters of the components of G_H.*

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[3] G. Frobenius and I. Schur, *Über die reellen Darstellungen der endlichen Gruppen*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1906 (I), pp. 186–208, especially p. 197. The author is indebted to G. de B. Robinson for referring him to this article.

[4] J. S. Frame, Proceedings of the National Academy of Sciences, loc. cit.

[5] A study of these matrices was made by I. Schur, *Zur Theorie der einfach transitiven Permutationsgruppen*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1933, pp. 598–623.

[6] This relation is well known in group theory. See, for example, W. Burnside, *Theory of Groups*, 1911, p. 275.

[7] J. S. Frame, Duke Mathematical Journal, loc. cit.

In §5 a study of the bilinear invariants of the components of G_H which are irreducible in the real domain leads to an alternate proof of Theorem A, based on methods similar to those of §4.

2. **Elementary properties of double cosets.** The set of elements permutable with a coset HS_i form a group N_i called the normalizer of the coset, whose intersection with H is a group D_i of order d_i leaving HS_i fixed in the permutation group G_H. We let D_iH_{ij} be the $k_i=h/d_i$ cosets of H with respect to D_i and note that for each j the elements $HS_tD_iH_{ij}=H(S_tH_{ij})$ form a different coset HS_j associated with HS_t. Hence there are k_t distinct cosets each occurring d_t times in the weighted double coset HS_tH. *If H is invariant in G,* then each $D_t=H$, each $k_t=1$, and *each coset is a double coset.* It is convenient to select the S_j for associated cosets so that $S_iH_{ij}=S_j$ and then to define $H_{ji}=S_j^{-1}S_i=H_{ij}^{-1}$, when obtaining the cosets of H with respect to its subgroup $D_j=H_{ij}^{-1}D_iH_{ij}$, which is permutable with HS_j. It is seen immediately that the subgroups $D_i, D_j, \cdots$ which are permutable with the right cosets $HS_i, HS_j, \cdots$ of a double coset HS_tH are all of the same order d_t and form a complete set of conjugate subgroups of H.

The following four properties of double cosets are simple enough to be given here without proof.[8]

THEOREM 2.1. *Each element of G lies in one and only one double coset of G with respect to a given subgroup H.*

THEOREM 2.2. *A double coset may be generated by any one of its elements.*

THEOREM 2.3. *The inverses $H_j^{-1}S_i^{-1}H_i^{-1}$ of the elements $H_iS_tH_j$ of the double coset K_t form a double coset $K_{t'}$ generated by S_i^{-1}, which has the same density as K_t.*

If $K_t=K_{t'}$, the double coset is called self-inverse.

THEOREM 2.4. *A double coset which contains a self-inverse element is self-inverse. In particular the double coset $H=K_1$ is self-inverse.*

The next three theorems show that the elements of a class of conjugates, of a left coset, and of the set of inverses of a right coset, are equally distributed among the right cosets of their double coset.

THEOREM 2.5. *Each coset of a double coset K_t contains the same num-*

[8] Several of the theorems in §2 are implied in the discussion of cosets in the standard texts on group theory. Nowhere has the author found them collected as properties of double cosets as such. They are stated here for convenience of reference for the later proofs.

ber $g_{\Omega\lambda}$ of the g_λ elements forming the class C_λ of conjugate elements in G.[9]

PROOF. Let $H_\theta S_i$ be a common element of HS_i and C_λ, and let the subgroup of D_i which is permutable with the element $H_\theta S_i$ be of index $g_{\Omega\lambda}$ in D_i. Then there will be $g_{\Omega\lambda}$ distinct elements each counted $d_i/g_{\Omega\lambda}$ times among the elements $\delta_\kappa^{-1}(H_\theta S_i)\delta_\kappa$, δ_κ in D_i. Each of these distinct elements of HS_i is transformed by H_{ij} into a different element of HS_j, so the numbers $g_{\Omega\lambda}$, $g_{\Omega\lambda}$, $\cdots$ are the same for all the cosets of K_t and may be denoted by $g_{\Omega\lambda}$.

THEOREM 2.6. *The h elements of the left coset $(H_\theta S_i)H$ are equally distributed among the k_i right cosets associated with HS_i, just d_i elements lying in each right coset.*

PROOF. Each element of H can be written uniquely in the form $\delta_\kappa H_{ij}$, where δ_κ is in D_i. For each j we obtain d_i elements $H_\theta S_i \delta_\kappa H_{ij} = (H_\theta S_i \delta_\kappa S_i^{-1})S_j$ in HS_j.

If we note further that the inverses of the elements of a right coset HS_i form a left coset $S_i^{-1}H$, then we obtain the following:

THEOREM 2.7. *The inverses of the elements of a right coset are equally distributed among the k_i cosets of the inverse double coset, with d_i elements in each. Furthermore each coset of a self-inverse double coset K_t contains just d_t of its own inverses.*

THEOREM 2.8. *Each element of the normalizer N_1 of H transforms a double coset K_t into itself or into a double coset $N_1^{-1}K_tN_1$ having the same distribution of elements among the classes C_λ as K_t does.*

These double cosets may be called conjugate double cosets.

PROOF. Each element is transformed into a conjugate element in the double coset generated by $N_1^{-1}S_tN_1$.

THEOREM 2.9. (a) *The number of times, k_{tuv}, that the identity element E occurs among the h^3 elements $HS_tHS_uHS_v$ depends only on the double cosets to which S_t, S_u, and S_v belong.*

(b) *The constant k_{tuv} is unchanged by an even permutation of the double cosets K_t, K_u, K_v, or by an odd permutation coupled with a change to inverse double cosets.*

(c) *The number of the h elements S_uHS_v which lie in the double coset $K_{t'}$ is k_{tuv}/d_t, and $\sum_{t=1}^{r} k_{tuv}/d_t = h$.*

(d) *For $t=1$, we have $k_{1uv} = k_{uv1} = hd_u\delta_{uv'}$.*

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[9] D. E. Littlewood, *Theory of Group Characters*, 1940, p. 149. Littlewood shows, if we change his notation to ours, that the characters in G_H of the elements of C_λ are $ng_{\Omega\lambda}/g_\lambda$.

(e) *The product of the double cosets K_tK_u is given by the equation*

$$K_tK_u = (h/(d_td_u))\sum_v k_{tuv}K_v. \tag{2.10}$$

To prove the various parts of Theorem 2.9, let us assume that H_a, H_b, and H_c are elements of H which satisfy the equation $H_aS_tH_bS_uH_cS_v=E$. The equation is still valid if we permute the six elements cyclicly, or if we interpose between two adjacent ones the product of an element of H and its inverse, thus replacing an element by another element in its double coset, or if we replace all six elements by their inverses in the opposite order. The remainder of the proof depends on eliminating the weighting factors from the weighted double cosets HS_tH, etc., but may be left to the reader.

3. **The number of self-inverse double cosets.** We shall now apply the theorems of §2 to obtain expressions for the number of self-inverse double cosets of a group.

THEOREM 3.1. *The number N_H of self-inverse double cosets K_t of G with respect to H is $1/g$ times the number of solutions of the equation*

$$S_jR^2 = H_kS_j, \qquad j = 1, 2, \cdots, n;\ H_k \text{ in } H,\ R \text{ in } G.$$

We rewrite the equation in the form $H_k^{-1}(S_jRS_j^{-1}) = (S_jRS_j^{-1})^{-1}$, which states that the inverse of the element $S_jRS_j^{-1}=T$ belongs to its own right coset. For each of the k_t cosets of K_t there are d_t available values of T, by Theorem 2.7. For each of these $h=d_tk_t$ values of T we may choose n generators S_j to determine an element $R=S_j^{-1}TS_j$ conjugate to T. Thus for each self-inverse double coset we obtain $g=nh$ solutions of the given equation, and the theorem is proved.

Now the permutation matrix corresponding to R^2 in G_H has a unit in the principal diagonal for every right coset which remains fixed by R^2, that is, for every solution of the equation $S_jR^2=H_kS_j$, H_k in H, $j=1, 2, \cdots, n$. Its trace is the number of these solutions for fixed R. When we sum these for all R in G and apply Theorem 3.1, we obtain the following theorem.

THEOREM 3.2. *The number N_H of self-inverse double cosets of G with respect to H is given by the formula*

$$N_H = (1/g)\sum_{R \text{ in } G} \chi(R^2), \tag{3.3}$$

where $\chi(S)$ is the trace of the matrix S in the permutation group G_H.

This theorem may be applied to the results of a paper by Frobenius

and Schur[10] to obtain the fundamental Theorem A of §1. It is known in the theory of group characters that for an irreducible representation Γ_i with character $\chi^i(R)$ the quantity

$$\sum_R \{[\chi^i(R)]^2 + \chi^i(R^2)\}/(2g) \tag{3.4}$$

is 1 or 0 according as Γ_i has or has not a symmetric bilinear invariant, and that

$$\sum_R \{[\chi^i(R)]^2 - \chi^i(R^2)\}/(2g) \tag{3.5}$$

is 1 or 0 according as Γ_i has or has not an alternating bilinear invariant. Following Frobenius[10] we set $c_i=1$, -1, or 0 according as Γ_i has a symmetric, an alternating, or no bilinear invariant. We shall call these respectively "symmetric," "quaternion," and "rotary" representations. Then

$$(1/g)\sum_R \chi^i(R^2) = c_i. \tag{3.6}$$

Now let G_H be reduced into irreducible components Γ_i with multiplicities μ_i^H. Then

$$G_H = \sum_i \mu_i^H \Gamma_i, \qquad \chi(R) = \sum_i \mu_i^H \chi^i(R). \tag{3.7}$$

Hence, by (3.3), (3.7), and (3.6), we have

$$N_H = (1/g)\sum_R \chi(R^2) = (1/g)\sum_R \sum_i \mu_i^H \chi^i(R^2) = \sum_i c_i \mu_i^H. \tag{3.8}$$

This formula is equivalent to Theorem A of §1.

4. **A theorem derived from a unitary reduction of a group and its double cosets.** When a group G is represented in two ways as a regular permutation group of degree g, using cosets with respect to the identity subgroup E, the one G_E obtained by the right multiplication of cosets by the elements of G, and the other G_E' by left multiplication, the matrices of G_E' form a basis for all matrices permutable with those of G_E, and vice versa. In that case each of the elements of G forms a double coset. But when cosets are taken with respect to a subgroup $H \neq E$, a basis for the matrices permutable with those of the right multiplication permutation group G_H is found in the double cosets of the left multiplication group. The matrices V_t described in §1 are

[10] G. Frobenius and I. Schur, loc. cit.

those of a complete set of invariant Hermitian forms of G_H on which all others are linearly dependent. For let $x_i, x_i = x(HS_i), i = 1, 2, \cdots, n$, be the variables permuted by G_H. Then the matrix R of G_H transforms the product $\bar{x}(H)x(HS_t)$ into $\bar{x}(HR)x(HS_tR)$, and if we write $R = H_jG_k$ and sum over all R in G, we obtain an Hermitian form invariant under the right multiplications of G_H. Since the subgroup D_t of H leaves HS_t fixed, each term of this form will contain the factor d_t. Dividing by d_t and writing $K_t = HS_tH/d_t$, we obtain the simpler invariant Hermitian form $\sum_k \bar{x}(HG_k)x(K_tG_k)$, whose matrix we have denoted by V_t. The matrix V_1 is the unit matrix. Each row and column of V_t contains k_t 1's and the rest 0's, and the sum of all the matrices V_t is a matrix consisting entirely of 1's.[11] To the inverse double coset $K_{t'}$ corresponds the transposed matrix V_t', which will be denoted by $V_{t'}$. Self-inverse double cosets have symmetric matrices V_t.

Now let U be a unitary matrix which completely reduces the group G_H into its irreducible components Γ_i of degree n_i and multiplicity μ_i^H, so that all equivalent components of $U^{-1}G_HU$ are actually identical and so that the invariant Hermitian form for each component is a diagonal form with unit matrix E_i of degree n_i. Then for the set of μ_i^H equal components Γ_i we have $(\mu_i^H)^2$ linearly independent Hermitian forms whose matrices $\sum_{\alpha\beta} E_i^{\alpha\beta} \rho_t^{\alpha\beta}$ are obtained as the direct product of E_i with an arbitrary matrix $(\rho_i^{\alpha\beta})$ of degree μ_i^H. The r matrices $M_t = U^{-1}V_tU$, obtained by transforming the Hermitian invariants of G_H, must be expressible as linear combinations of the r matrices $E_i^{\alpha\beta}$. It is convenient to arrange the symbols $E_i^{\alpha\beta}$ in some arbitrary order starting with $E_1^{11} = E_1$, and to assign to each a single subscript γ. We write

$$M_t = \sum_\gamma E_\gamma \rho_{\gamma t}; \qquad \text{where } E_\gamma = E_i^{\alpha\beta} \quad \text{and } \rho_{\gamma t} = \rho_i^{\alpha\beta}(K_t); \quad \gamma, t = 1, 2, \cdots, r. \tag{4.1}$$

Since the matrices M_t and E_γ each form a basis for the invariant Hermitian forms of $U^{-1}G_HU$, the matrix $(\rho_{\gamma t})$ is nonsingular.

The M_t combine according to a rule similar to the rule of combination for the inverse double cosets $K_{t'}$, as given in 2.10, namely,

$$M_tM_u = \sum_\tau c_{tu\tau} M_\tau, \qquad c_{tu\tau} = k_{t'u'\tau}/(d_td_u). \tag{4.2}$$

Since this is also the rule of combination for the matrices $\rho_i^{\alpha\beta}(K_t)$ corresponding to a given Γ_i, these matrices give that representation for the left multiplication of double cosets which is associated with Γ_i.

[11] J. S. Frame, Duke Mathematical Journal, loc. cit.

Now when G_E and G_E' are simultaneously reduced, the component of G_E' on the same variables as the n_i components Γ_i of G_E consists of the direct product of E_i with a representation equivalent to Γ_i'. When the matrices of this representation are summed over all elements in each double coset, the resulting set of r matrices when reduced will contain the component of degree $\mu_i^H \leqq n_i$, expressible in terms of the variables of G_H, which is equivalent to the matrices $\rho_i^{\alpha\beta}(K_t)$. Hence the $\rho_i^{\alpha\beta}(K_t)$ are expressible in terms of the matrices of Γ_i'.

The rule (4.2) is also the rule of combination for the matrices V_t,[12] so the coefficients c_{tuv} must be integers, and the character of the product $M_{t'}M_u$ is given by the formula

$$\chi(M_{t'}M_u) = nc_{t'u1} = nk_t\delta_{tu}, \tag{4.3}$$

in view of Theorem 2.9 (d). Since U is unitary, the matrix $M_{t'}$ is the transposed conjugate $\overline{M}_t'$ of M_t. We shall denote by $E_{\gamma'}$ the transposed conjugate of E_γ, and by n_γ the degree of the corresponding irreducible representation.

Next consider the matrices $(M_{t'u})$ and $(E_{\gamma'\delta})$ of degree r whose elements are

$$M_{t'u} = \chi(M_{t'}M_u) = \chi(V_{t'}V_u) = nk_t\delta_{tu}, \tag{4.4}$$

$$E_{\gamma'\delta} = \chi(E_{\gamma'}E_\delta) = n_\gamma\delta_{\gamma\delta}. \tag{4.5}$$

We obtain a relation between $(M_{t'u})$ and $(E_{\gamma'\delta})$ as follows:

$$M_{t'u} = \chi\left(\sum_\gamma \bar{p}_{t\gamma}E_{\gamma'}\sum_\delta E_\delta\rho_{\delta u}\right) = \sum_{\gamma,\delta}\bar{p}_{t\gamma}E_{\gamma'\delta}\rho_{\delta u}. \tag{4.6}$$

Denoting the determinant of $(\rho_{\delta u})$ by P, we have, by (4.4), (4.5), (4.6),

$$\prod_{t=1}^{r}(nk_t) = \left(\prod_{\gamma=1}^{r}n_\gamma\right)\overline{P}P = \left(\prod_{i=1}^{r'}n_i^{(\mu_i^H)^2}\right)\overline{P}P. \tag{4.7}$$

Since $\sum_t\rho_{\gamma t}=n\delta_{\gamma 1}$, the determinant P may be written in the form nP_1, where P_1 is the minor of ρ_{11} in P. Factoring n^2 from both sides of (4.7) and using the notation of Theorem B, we have

$$n^{r-2}K = N\overline{P}_1P_1, \quad \text{where} \quad K = \prod_{t=1}^{r}k_t, \quad N = \prod_{i=1}^{r'}n_i^{(\mu_i^H)^2}. \tag{4.8}$$

Since for each i the matrices $\rho_i^\alpha(K_t)$ give a representation of the ring of matrices V_t in which the coefficients of combination are integers, the matrix P_1 is an algebraic integer. It belongs to the field of char-

[12] J. S. Frame, Duke Mathematical Journal, loc. cit.

acters of the Γ_i. This completes the proof of Theorem B. It will be noted that if the representations Γ_i have rational characters, which will certainly be true if their degrees are all distinct, then the quotient $n^{r-2}K/N$ will be a perfect square.

5. **A theorem derived from a real orthogonal reduction of a group and its double cosets.** Let us now consider the reduction of G_H by a real orthogonal matrix O into a form which is irreducible in the real domain. The component representations are of three types: (1) Those "symmetric" representations $\Gamma_i^{(+)}$ which are absolutely irreducible in the complex domain and have a symmetric bilinear invariant. (2) Those "quaternion" representations $2\Gamma_i^{(-)}$ which consist of two equivalent complex components of even degree each with real characters and each with an alternating but not a symmetric bilinear invariant. The pair together have a third alternating bilinear invariant and a symmetric bilinear invariant. The matrices of the four invariants, suitably normalized, combine like the quaternion units. (3) Those "rotary" representations $\Gamma_i^{(0)}+\Gamma_{i'}^{(0)}$ which have two non-equivalent conjugate complex absolutely irreducible components, each having complex characters, but having no bilinear invariant. Taken together they have an alternating and a symmetric bilinear invariant which, when suitably normalized, combine like the real and imaginary units.

The matrices $O^{-1}V_iO$ form a basis for the invariant bilinear forms, but in place of the other basis matrices E_γ used in §4, we now use matrices E_γ which for the symmetric representations of type 1 are like the old E_γ, for the quaternion representations of type 2 come in sets of four which multiply like the quaternion units, and for the rotary representations of type 3 come in pairs which multiply like the real and imaginary units. Those of type 2 are as follows:

$$(5.1)\qquad \begin{pmatrix} e_i & 0 & 0 & 0 \\ 0 & e_i & 0 & 0 \\ 0 & 0 & e_i & 0 \\ 0 & 0 & 0 & e_i \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & e_i & 0 \\ 0 & 0 & 0 & e_i \\ -e_i & 0 & 0 & 0 \\ 0 & -e_i & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -e_i & 0 & 0 \\ e_i & 0 & 0 & 0 \\ 0 & 0 & 0 & e_i \\ 0 & 0 & -e_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & e_i \\ 0 & 0 & -e_i & 0 \\ 0 & e_i & 0 & 0 \\ -e_i & 0 & 0 & 0 \end{pmatrix},$$

where e_i denotes the unit matrix of degree $n_i/2$.

We next normalize each of the matrices V_t and E_γ by dividing by the square root of the sum of the squares of its coefficients. From the new normalized matrices V_t^* and E_γ^*, thought of as of degree n, we form a new set of matrices of degree r defined by

$$\begin{aligned} V_{tu}^* &= \chi(V_t^* V_u^*), & V_{t'u}^* &= \chi(V_t^{*\prime} V_u^*), \\ E_{\gamma\delta}^* &= \chi(E_\gamma^* E_\delta^*), & E_{\gamma'\delta}^* &= \chi(E_\gamma^{*\prime} E_\delta^*). \end{aligned} \tag{5.2}$$

Then (V_{tu}^*) and $(E_{\gamma\delta}^*)$ are each the unit matrix, whereas $(V_{tu}^*) = (\delta_{t'u})$ is a matrix whose trace is the number N_H of self-inverse double cosets, and $(E_{\gamma\delta}^*)$ is a matrix whose trace is $\sum_{i=1}^{r'} c_i \mu_i^H$, since each component $\Gamma_i^{(+)}$ of type 1 contributes $+1$, each real component $2\Gamma_i^{(-)}$ of type 2 containing a pair of equivalent irreducible representations contributes $+1 -1 -1 -1 = -2$, and each real component of type 3 contributes $+1 -1 = 0$. Now let

$$O^{-1} V_t^* O = \sum_\gamma E_\gamma^* q_{\gamma t} \tag{5.3}$$

define a matrix Q which changes the basis of the bilinear forms of $O^{-1} G_H O$. Then

$$\begin{aligned} V_{tu}^* = \chi(V_t^* V_u^*) &= \chi(O^{-1} V_t^* O \cdot O^{-1} V_u^* O) = \chi\left(\sum_\gamma E_\gamma^* q_{\gamma t} \cdot \sum_\delta E_\delta^* q_{\delta u} \right) \\ &= \sum_{\gamma,\delta} q_{\gamma t} E_{\gamma\delta}^* q_{\delta u}. \end{aligned} \tag{5.4}$$

Similarly, since $(V_{t'u}^*)$ and $(E_{\gamma'\delta}^*)$ are unit matrices, we have

$$V_{t'u}^* = \sum_{\gamma,\delta} q_{\gamma' t'} E_{\gamma'\delta} q_{\delta u} = \sum q_{\delta t'} q_{\delta u} = \delta_{t'u}. \tag{5.5}$$

Hence Q is an orthogonal matrix, and (5.4) may be written in the form

$$(V_{tu}^*) = Q^{-1} (E_{\gamma\delta}^*) Q. \tag{5.6}$$

By equating the traces of (V_{tu}^*) and $(E_{\gamma\delta}^*)$ given above, an alternate proof of Theorem A is obtained.

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CONDITIONS FOR THE CONTINUITY OF ARC-PRESERVING TRANSFORMATIONS[1]

D. W. HALL AND W. T. PUCKETT, JR.

1. **Introduction.** A single-valued transformation $T(A)=B$, where A and B are topological spaces, is said to be *arc-preserving*[2] provided that the image of every simple arc in A is either a simple arc or a a single point in B. Even when A is a simple arc, an arc-preserving transformation may fail to be continuous; for example: on the unit interval $(x_0=0\leqq x\leqq 1=x_1)$ let $x_n=1/n$ $(n=1, 2, 3, \cdots)$. Define $T(x_0)=x_0$ and for each interval A_n $(x_{n+1}\leqq x\leqq x_n)$ let $T(A_n)=A$ be a topological transformation such that $T(x_n)=x_0$ or x_1 according as n is even or odd. Then the transformation $T(A)=A$ is arc-preserving, but fails to be continuous at x_0.

The results of this paper concern conditions under which an arc-preserving transformation is continuous, and the conclusions lead to homeomorphisms. We consider only the case where A is a locally connected continuum. The transformation T may be made continuous by putting conditions on the space A or by putting added conditions on the transformation T itself. In this paper we take both points of view. We shall say that A is *strongly arcwise connected* provided every infinite subset of A intersects some arc of A in infinitely many points. Our principal theorem states that if A is cyclic and $T(A)=B$ is arc-preserving then T will be topological or B will be an arc provided either A is strongly arcwise connected or T is tree-preserving[3] (that is, the image of every tree in A is a tree or a single point in B). Moreover, we show that if B is not an arc then A must be strongly arcwise connected in order that a topological mapping be the only arc-preserving transformation of A onto B.

Throughout the paper A is a locally connected continuum and T is a single-valued transformation, but not necessarily continuous. It is understood that a single point is to be regarded as an arc.

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[1] Presented to the Society in parts as follows: April 6, 1940, under the title *On arc-preserving transformations*, by Puckett; April 26, 1940, under the title *On arc and tree preserving transformations*, by D. W. Hall; and September 12, 1940, under the title *Arc-preserving transformations of a certain class of spaces*, by Hall and Puckett.

[2] See G. T. Whyburn, *Arc-preserving transformations*, American Journal of Mathematics, vol. 58 (1936), pp. 305–312. See also D. W. Hall and G. T. Whyburn, *Arc- and tree-preserving transformations*, Transactions of this Society, vol. 48 (1940), pp. 63–71.

[3] See R. G. Simond, Duke Mathematical Journal, vol. 4 (1938), pp. 575–589; also Hall and Whyburn, loc. cit.

2. **On strongly arcwise connected sets.** The set A is said to be *strongly arcwise connected* provided that every infinite subset of A contains infinitely many points which lie on an arc in A. The following are immediate consequences of this definition:

(2.1) *A strongly arcwise connected set is compact and locally connected.*

(2.2) *In order that a continuum be strongly arcwise connected it must be the sum of a finite number of cyclic chains.*

(2.3) *The property of a continuum's being strongly arcwise connected is cyclicly reducible.*[4]

(2.4) *The property of being strongly arcwise connected is invariant under an arc-preserving transformation.*

(2.5) *If A is strongly arcwise connected and $T(A)=B$ is a one-to-one arc-preserving transformation, then T is topological.*

PROOF. We need only show that $T(A)=B$ is continuous. To this end let $\{x_n\}$ be a sequence of points converging to a point x in A. Since A is strongly arcwise connected we lose no generality in assuming that all the points $\{x_n\}$ lie on an arc α in A. For each n let α_n be the irreducible subarc of α containing $x+\sum_{i=n}^{\infty}x_i$; then $\prod\alpha_n=x$. Now $T(\prod\alpha_n)=\prod T(\alpha_n)$, since T is one-to-one. Thus $T(x)=\prod T(\alpha_n)$ is the intersection of a monotone decreasing sequence of arcs. Thus $T(x_n)$ converges to $T(x)$, since $T(x_n)$ is contained in $T(\alpha_n)$ for every n.

The following example shows that condition of strong arcwise connectivity cannot be omitted:

EXAMPLE. *There exists a one-to-one arc-preserving transformation $T(A)=B$, where A is a cyclicly connected continuum, which is not continuous.*

PROOF. The example will be constructed in the euclidean plane. Let L be the unit interval and for every positive integer n let A_n be a line segment of length $1/n^2$ erected perpendicular to L at the point $1/n$. Define a_n as the end of A_n not on the line L and let B_n be the segment joining the point a_n to the origin. Then the cyclicly connected continuum of the example will consist of the unit interval L together with all the segments A_n and B_n.

To construct B let O denote the origin and for every positive integer n define Q_n as the point $(1/n, 0)$. Let M denote the line $x=.7$, and $P_1=(1, 1)$. We may then define an infinite sequence of points

[4] See Kuratowski and Whyburn, Fundamenta Mathematicae, vol. 16 (1930), pp. 305–331.

$\{P_n\}$ as follows: assuming that P_{n-1} has been defined let R_{n-1} be the point of intersection of the bisector of the angle $OP_{n-1}Q_{n-1}$ with the line M, and define P_n as the midpoint of the segment $P_{n-1}R_{n-1}$. Let C_n be the segment OP_n and D_n the segment P_nQ_n. The set B is then defined as the sum of L and all the line segments C_n and D_n.

The transformation $T(A)=B$ is now easily set up. Define $T(x)=x$ for all x in L, and let T send the sum of A_n and B_n topologically into the sum of C_n and D_n. Evidently $T(A)=B$ as thus defined is one-to-one and arc-preserving, but not continuous at the origin.

(2.6) *Any locally connected continuum A which is not strongly arcwise connected may be mapped onto the unit circle by an arc-preserving transformation.*

PROOF. Since A is not strongly arcwise connected it contains a sequence of disjoint regions[5] $\{U_n\}$ such that no arc in A intersects infinitely many of the U_n. Clearly, the set $M=A-\sum U_n$ and the points of $\sum U_n$ give an upper semi-continuous decomposition of A. This decomposition determines a continuous transformation $T_1(A)=A_1$ such that each $T_1(U_n)$ is a component of $A_1-T_1(M)=A_1-p_1$ and $A_1=p_1+\sum T_1(U_n)$. Now for each positive integer n let L_n be the line segment in the euclidean plane between the points $x_0=(0, 0)$ and $x_n=(1/n, 1/n^2)$ and define $A_2=\sum L_n$. Then there exists a continuous transformation $T_2(A_1)=A_2$ such that $T_2(\overline{T_1(U_n)})=T_2(T_1(U_n)+p_1)=L_n$. Finally, let B be the unit circle $x=\cos\theta$, $y=\sin\theta$ $(0\leqq\theta<2\pi)$ and let B_n be the subarc of B given by $0\leqq\theta\leqq(2n-1)\pi/n$. A sequence of topological transformations $T_3(L_n)=B_n$ which in every case maps x_0 onto the point $(1, 0)$ of B defines a transformation $T_3(A_2)=B$. Let α be any arc of A and consider its image $T(\alpha)=T_3T_2T_1(\alpha)$ in B. Since $T_2T_1(A)=A_2$ is continuous and α can intersect at most a finite number of the regions U_n, it follows that $T_2T_1(\alpha)$ is a connected subset of A_2 and is contained in some $\sum_{i=1}^{k}L_{n_i}$ $(n_{i+1}>n_i)$. Moreover, by construction $T_3(\sum_{i=1}^{k}L_{n_i})=B_{n_k}$ is continuous and, consequently, $T_3T_2T_1(\alpha)$ is an arc. Thus $T(A)=T_3T_2T_1(A)=B$ is arc-preserving. However, it will be noted that T fails to be continuous at any point in the limit superior of $\{U_n\}$, but is continuous on every arc α of A.

3. **Lemmas.** In this section we obtain some preliminary results concerning arc-preserving transformations defined on cyclic locally connected continua. The set B is a topological space.

(3.1) *Let A be cyclic, let $T(A)=B$ be arc-preserving, and let $G=[\alpha]$*

[5] See Hall and Puckett, *Strongly arcwise connected spaces*, to appear in American Journal of Mathematics.

be the set of all arcs with endpoints in $T^{-1}(p')$ and $T^{-1}(q')$ respectively. If the image of every simple closed curve of A is an arc, then $\prod T(\alpha)$ contains an arc joining p' and q'.

PROOF. Let $G^* = [\alpha^*]$ be the subcollection of arcs of G which have $p^* \in T^{-1}(p')$ and $q^* \in T^{-1}(q')$ for their endpoints, and let α_1^* and α_2^* be any two arcs of this collection. Now let β_1' and β_2' be subarcs of $T(\alpha_1^*)$ and $T(\alpha_2^*)$ respectively which have p' and q' as endpoints. Suppose there exists a point x' of β_1' not contained in β_2'; then $\beta_1' + \beta_2'$ contains a simple closed curve J'. Therefore, since $T^{-1}(x')$ is disjoint with α_2^*, $\alpha_1^* + \alpha_2^*$ contains a simple closed curve J such that $T(J)$ contains J', contrary to hypothesis. Define $\beta' = \beta_1' = \beta_2'$, and suppose there exists a point $p \neq p^*$ of $T^{-1}(p')$. Since A is cyclic it contains an arc $p^*q^* + q^*p$. Now $T(p^*q^*)$ contains β', since p^*q^* is an arc of G^*. Moreover, $T(q^*p)$ must contain β', for otherwise $T(p^*q^* + q^*p)$ would contain a simple closed curve. Because of the symmetry of the above argument it follows that $\prod T(\alpha)$ contains β'.

(3.2) *If $T(A) = B$ is arc-preserving and J is a simple closed curve, then $T(J) = J'$ is topological or J' is an arc.*

PROOF. Suppose J' is not an arc. Then, since J is strongly arcwise connected, we need only show that $T(J) = J'$ is one-to-one, by virtue of (2.5). If $T(J) = J'$ is not one-to-one there exist two points x and y of J such that $T(x) = T(y)$. Express $J = \alpha + \beta$, where α and β are arcs such that $\alpha \cdot \beta = x + y$. There exist in α distinct points p and q whose images are the endpoints of $T(\alpha)$. (In case $T(\alpha)$ is degenerate these points may be x and y.) Let the points be so named that $\alpha = xp + pq + qy$, where any two of the arcs on the right have at most a common endpoint. Now the endpoints of $T(\alpha)$ lie in its subarcs $T(xp)$ and $T(qy)$, which have a common point. Consequently, $T(xp + qy) = T(xp) + T(qy) = T(\alpha)$. Thus the arc $\gamma = px + \beta + qy$ is such that $T(\gamma) = J'$, contrary to the hypothesis that J' is not an arc.

(3.3) *If $T(A) = B$ is arc-preserving, where A is cyclic, and if there exists a simple closed curve J in A such that $T(J) = J'$ is not an arc, then T is one-to-one on A.*

PROOF. Since J' is not an arc, it follows from (3.2) that $T(J) = J'$ is topological. Let z be any point of J and suppose $z \neq T^{-1}T(z)$. Then there exists a point z_1 in $A - J$ such that $T(z_1) = T(z)$. Let cz_1d be an arc in A spanning[6] J, and suppose $d \neq z$. Write J as the sum of two simple arcs α and β having precisely the points z and d in common.

[6] An arc axb is said to *span* a point set M provided $M \cdot axb = a + b$.

Now $T(z)=T(z_1)$ and T is topological on α. Consequently, since $T(z_1d+\alpha)$ is an arc, we have $T(z_1d)$ contains $T(\alpha)$. Therefore, $T(z_1d+\beta)$ contains J', contrary to the fact that J' is not an arc. Hence for every point z of J we have $z=T^{-1}T(z)$.

Now let z be a point of $A-J$ and let α be an arc through z spanning J and dividing J into two arcs β and γ. From the above and (3.2) it follows that $T(\alpha+\beta)$ is a simple closed curve. Consequently, $z=T^{-1}T(z)$ and, therefore, T is one-to-one on A.

(3.4) *Under the hypotheses of* (3.3) *either of the following conditions suffices to make T topological*: (a) *A is strongly arcwise connected, or* (b) *T sends trees into compact sets.*

PROOF. That (a) suffices is immediate from (2.5). To show that (b) suffices we need only establish the continuity of T. Assume T is not continuous. Then there exists a sequence of points $\{x_n\}$ converging to a point x of A such that either the sequence $T(x_n)=x_n'$ converges to a point $y'\neq T(x)$ or the set $T(x_n)$ has no limit point. Now there exists a tree t containing infinitely many of the x_n, but not $y=T^{-1}(y')$ if y' exists. It follows in either case that $T(t)$ is not compact.[7] This contradiction completes the proof.

(3.5) *Let J be any simple closed curve in the cyclic continuum A and let $T(A)=B$ be arc-preserving but not topological. Then either of the following conditions suffices to make $T(J)=J'$ a free arc of B*: (a) *A is strongly arcwise connected, or* (b) *the image of each tree in A is a locally connected continuum.*

PROOF. From (3.2) we see that $T(J)=J'$ is topological or J' is an arc. If $T(J)=J'$ is topological, then by (3.4) we see that $T(A)=B$ is topological and the theorem is established. Hence assume that J' is an arc $a'x'b'$ of B which is not a free arc of B. To obtain the desired contradiction we first establish the following assertion:

(i) *There exists an arc uv in A such that $T(uv)$ contains a nondegenerate subarc $u'v'$ having exactly the point v' in common with J', where v' is an interior point of the arc J'.*

To prove (i) we observe that since J' is not a free arc of B there must exist a sequence of points x_n' of $B-J'$ converging to an interior point x' of J'. If A is strongly arcwise connected there exists an arc N in A intersecting infinitely many of the sets $T^{-1}(x_n')$. Then $T(N)$ contains a nondegenerate subarc $u'v'$ satisfying the conditions of (i) and

[7] An arc α is a *free* arc of M provided α spans $\overline{M-\alpha}$.

we immediately obtain the arc uv as a subarc of N joining a point of $T^{-1}(u')$ to a point of $T^{-1}(v')$. This proves (i) under our first hypothesis. Assume next that (b) holds. Then there exists a tree t in A intersecting $T^{-1}(x')$ and infinitely many of the sets $T^{-1}(x_n')$. Thus $T(t)$ is a locally connected continuum in B containing x' and infinitely many of the points x_n'. Hence $T(t)$ is locally arcwise connected and thus contains an arc $u'v'$ satisfying (i). The arc uv is then obtained as any arc in t joining a point of $T^{-1}(u')$ to a point of $T^{-1}(v')$. This completes the proof of (i).

Now since A is cyclic there exists an arc H in A intersecting $T^{-1}(u')$ and having its endpoints in $T^{-1}(a')$ and $T^{-1}(b')$. It follows at once from (3.1) that $T(H)$ contains J'. Now $T(H)$ contains both u' and v' and hence a subarc joining these points. Since $T(H)$ is an arc, this subarc must contain either a' or b', hence we assume that it contains a' and denote it by $u'a'v'$. From (3.1) it follows that if M is any arc in A having its endpoints in $T^{-1}(u')$ and $T^{-1}(v')$ then $T(M)$ contains $u'a'v'$. But this tells us at once that the image of the arc uv given by (i) must contain a simple closed curve. This contradiction completes the proof.

4. **Principal theorem.** We shall now prove our principal theorem.

(4.1) *Let $T(A)=B$ be arc-preserving, where A is a cyclic locally connected continuum and B is not an arc. Then T is topological if either A is strongly arcwise connected or T is tree-preserving.*

PROOF. Assume T is not topological. Then by (3.5) the image of every simple closed curve J of A is a free arc of B and, consequently, every two points of B lie on a free arc of B. Thus to show that B is a simple closed curve it is certainly sufficient to show that B is a locally connected continuum. This follows at once from (2.1) and (2.4) if A is strongly arcwise connected. Hence we need only establish it in the case where T is tree-preserving. That B is compact follows at once from this condition since every convergent sequence of points in A lies on a tree in A and the image of this tree is a tree. Assuming B not locally connected we can find two points a and b in B at which B fails to be locally connected. Let axb be a free arc in B and d a point of B not on this free arc. Then there exists a free arc dx in B where x is an interior point of axb. Thus either a or b is interior to the free arc dx of B which is impossible since B is not locally connected at either a or b. Hence we have established the fact that under either of our hypotheses B must be a simple closed curve.

We show first that if z' is any point of B there exists an arc β of A

such that z' is an interior point of $T(\beta)$. Since B is a simple closed curve, it contains an arc $p'z'q'$ having z' as an interior point. Let $\{p_n'\}$ be a sequence of points in the arc $p'z'$ converging monotonically to z'. Let t be a tree in A intersecting infinitely many of the sets $T^{-1}(p_n')$. If T is tree-preserving then $T(t)$ is an arc $t'=a'b'$. There exists in t an arc α intersecting both $T^{-1}(a')$ and $T^{-1}(b')$. Since $T(\alpha)=t'$, it follows that $T(\alpha)$ must contain a subarc $\gamma_p=z'x'$ of $z'p'$. If A is strongly arcwise connected the same result may be obtained by taking α as an arc in A which intersects infinitely many of the sets $T^{-1}(p_n')$. Likewise we obtain an arc $\gamma_q=z'y'$ which is a subarc of both $z'q'$ and the image of an arc of A. Let β be an arc in A intersecting $T^{-1}(x')$, $T^{-1}(y')$, and $T^{-1}(z')$. By virtue of (3.1), $T(\beta)$ contains $\gamma_p+\gamma_q$, an arc which has z' as an interior point.

It follows from the above and the Heine-Borel theorem that there exists a finite number of arcs $\alpha_1, \alpha_2, \cdots, \alpha_n$ $(n\geqq 2)$ in A such that $\sum_{i=1}^{n} T(\alpha_i)=\sum\alpha_i'=B$. Moreover, the α_i may be so selected and named that $\alpha_i'\cdot\alpha_k'$ is empty except for $k=i-1, i, i+1$ $(n+1=1)$. Let p_i' and q_i' be the endpoints of α_i', and assume them so named that p_{i+1}' is a point of α_i' while q_{i+1}' is not. Select in A an arc $\beta_1=p_1q_1+q_1q_2$, where the points p_1, q_1, and q_2 are in $T^{-1}(p_1')$, $T^{-1}(q_1')$, and $T^{-1}(q_2')$ respectively. By (3.1), $T(p_1q_2)$ contains α_1', and consequently a point p_2 of $T^{-1}(p_2')$. The subarc p_2q_2 of β_1 has an image which contains α_2', by (3.1). Therefore $T(\beta_1)$ contains $\alpha_1'+\alpha_2'$. To complete the induction assume that an arc β_r has been obtained in A such that $T(\beta_r)$ contains $\alpha_1'+\alpha_2'+\cdots+\alpha_{r+1}'$. Select in A an arc $\beta_{r+1}=p_1q_{r+1}+q_{r+1}q_{r+2}$, where the points p_1, q_{r+1}, and q_{r+2} are in $T^{-1}(p_1)$, $T^{-1}(q_{r+1})$, and $T^{-1}(q_{r+2})$ respectively. By applying (3.1) as above it follows that $T(p_1q_{r+1})$ contains $\alpha_1+\cdots+\alpha_{r+1}$, and, finally, that $T(\beta_{r+1})$ contains $\alpha_1+\cdots+\alpha_{r+2}$. Therefore, by induction, there exists an arc β_{n-1} in A such that $T(\beta_{n-1})=B$, which, as a consequence of our supposition, is a simple closed curve. This contradiction completes the proof of the theorem.

(4.11) COROLLARY. *Let A be a locally connected continuum having no local separating points. If $T(A)=B$ is arc-preserving, then either B is an arc or T is topological.*

PROOF. Since A has no local separating point it can contain no cut point. Moreover, A is strongly arcwise connected, since every closed and totally disconnected set of A is contained in an arc.[8] Consequently, A satisfies the hypotheses of (4.1).

[8] See G. T. Whyburn, *On disconnected sets*, Fundamenta Mathematicae, vol. 18 (1931), pp. 48–60.

The following theorem is an immediate consequence of (4.1) and (2.6):

(4.2) *The class of cyclic strongly arcwise connected continua consists exactly of all cyclic locally connected continua A such that every arc-preserving transformation $T(A)=B$, where B is not an arc, is topological.*

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A NOTE ON SUBGEOMETRIES OF PROJECTIVE GEOMETRY AS THE THEORIES OF TENSORS[1]

T. L. WADE

Klein's viewpoint (A) of a geometry as the invariant theory of a transformation group, as formulated in the Erlanger Programm in 1870,[2] has played an important part in the study of geometry during the past half century. A number of explicit utilizations of this viewpoint in invariant aspects of algebraic geometry have been made.[3] In the last decade the viewpoint (B) of a geometry as the theory of a tensor has received considerable theoretical discussion and utilization in connection with the new differential geometries.[4] While the adjunction argument, whereby subgeometries of projective geometry result from the latter by holding certain forms latent, has had considerable use,[5] and is closely related to tensor algebra, there seems to have been no explicit treatment of algebraic invariants for subgeometries of projective geometry from the viewpoint (B) with the use of tensor algebra. To indicate how this might be done is the purpose of this paper. The material here is largely an application and continuation of the basic paper by Cramlet.[6]

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[1] Presented to the Society, April 27, 1940.

[2] F. Klein, *Gesammelte Mathematische Abhandlungen*, Berlin, 1921, vol. 1, p. 460.

[3] C. C. MacDuffee, *Euclidean invariants of second degree curves*, American Mathematical Monthly, vol. 33 (1926), pp. 243–252; *Covariants of r-parameter groups*, Transactions of this Society, vol. 39 (1933).

[4] J. A. Schouten and J. Haantjes, *On the theory of the geometric object*, Proceedings of the London Mathematical Society, vol. 42 (1937), pp. 356–376.

[5] H. Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton University Press, 1939, pp. 254–258; H. W. Turnbull, *The Theory of Determinants, Matrices, and Invariants*, Blackie and Son, 1929, chap. 21.

[6] C. M. Cramlet, *The derivation of algebraic invariants by tensor algebra*, this Bulletin, vol. 34 (1928), pp. 334–342.

If in the $(n-1)$ projective geometry we hold the linear form L_iX^i, where $L_i=(0, \cdots, 0, 1)$ latent, we have the special numerical covariant tensor L_i. Then from Theorems 1 and 2 of Cramlet we have the following theorem:

THEOREM 1. *Affine geometry as a subgeometry of projective geometry is the theory of the tensor* L_i.

Similarly, U_i being dual covariant coordinates, holding the quadratic form

$$E^{ij}U_iU_j = 0$$

latent, where

$$E^{ij} = \begin{bmatrix} 1 & 0 & 0 \cdots 0 & 0 \\ 0 & 1 & 0 \cdots 0 & 0 \\ \cdot & \cdot & \cdots\cdots & \cdot \\ 0 & 0 & 0 \cdots 1 & 0 \\ 0 & 0 & 0 \cdots 0 & 0 \end{bmatrix},$$

we have a second theorem:

THEOREM 2. *Euclidean geometry as a subgeometry of affine geometry is the theory of the tensor* E^{ij}, *and as a subgeometry of projective geometry is the theory of the tensors* L_i *and* E^{ij}.

THEOREM 3. *Every euclidean concomitant for a set of ground forms in* X^i *and dual variables* U_i, *where* $X^iU_i=0$, *is expressible by composition as a tensor of order zero with the use of the coefficient tensors of the ground forms and the tensors* $\epsilon^{i_1\cdots i_n}$, $\epsilon_{i_1\cdots i_n}$, L_i, E^{ij}, X^i, *and* U_i. *The first two of these tensors are the commonly used skew symmetrical numerical tensors.*

In applications it is advantageous to introduce the tensor $A^{i_1\cdots i_{n-1}}$ where $A^{i_1\cdots i_{n-1}} = \epsilon^{i_1\cdots i_n}L_{i_n}$. For $n=3$,

$$A^{ij} = \epsilon^{ijk}L_k = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

As an illustration of an application we give a corallary:

COROLLARY. *An algebraically complete system of euclidean invariants for the ternary cubic curve*

$$C_{ijk}X^iX^jX^k = 0, \qquad i, j, k = 1, 2, 3; C_{ijk} = C_{ikj} = C_{jik},$$

consists of the seven invariants

$$I_1 = C_{a_1b_1c_1}C_{a_2b_2c_2}A^{a_1a_2}A^{b_1b_2}E^{c_1c_2},$$

$$I_2 = C_{a_1a_2c_1}C_{b_1b_2c_2}E^{a_1a_2}E^{b_1b_2}E^{c_1c_2},$$

$$I_3 = C_{a_1b_1c_1}C_{a_2b_2d_1}C_{a_3c_2d_2}\epsilon^{a_1a_2a_3}A^{b_1b_2}A^{c_1c_2}A^{d_1d_2},$$

$$I_4 = C_{a_1b_1e_1}C_{a_2b_2f_1}C_{c_1d_1e_2}C_{c_2d_2f_2}A^{a_1a_2}A^{b_1b_2}A^{c_1c_2}A^{d_1d_2}A^{e_1e_2}A^{f_1f_2},$$

$$I_5 = C_{a_1b_1c_1}C_{b_2c_2d_1}C_{c_3d_2a_2}C_{d_3a_3b_2}\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}\epsilon^{d_1d_2d_3},$$

$$I_6 = C_{a_1b_1c_1}C_{a_2b_2d_1}C_{b_3c_2e_1}C_{c_3a_3f_1}C_{d_2e_2f_2}C_{d_3e_3f_3}$$
$$\cdot\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}\epsilon^{d_1d_2d_3}\epsilon^{e_1e_2e_3}\epsilon^{f_1f_2f_3},$$

$$I_7 = C_{a_1b_1c_1}C_{d_1e_1f_1}A^{a_1d_1}\epsilon^{b_1e_1g_1}E^{c_1f_1}$$
$$\cdot C_{a_2b_2c_2}C_{d_2e_2f_2}A^{a_2d_2}\epsilon^{b_2e_2g_2}E^{c_2f_2}$$
$$\cdot C_{a_3b_3c_3}C_{d_3e_3f_3}A^{a_3d_3}\epsilon^{b_3e_3g_3}E^{c_3f_3}C_{g_1g_2g_3}.$$

In expanded form

$$I_1 = 2[C_{111}C_{122} - (C_{112})^2 - (C_{122})^2 + C_{112}C_{222}],$$

$$I_2 = (C_{111} + C_{122})^2 + (C_{112} + C_{222})^2,$$

$$I_3 = 6\begin{vmatrix} C_{111} & C_{112} & C_{113} \\ C_{211} & C_{221} & C_{231} \\ C_{221} & C_{222} & C_{223} \end{vmatrix}.$$

The expansions of the other invariants are considerably longer; I_5 contains twenty-five terms and I_6 one hundred and three terms. These seven invariants are, except in some cases for a constant factor, the members of the algebraically complete system of euclidean invariants for the cubic curve found through geometric means by Thomae.[7]

An invariant study of the geometry associated with the Galilei-Newton group of ten parameters has been made by Weitzenböck using the symbolic notation.[8] In the Galilei-Newton geometry, with X^i $(i=1, \cdots, 5)$ the homogeneous coordinates of a point, the basic latent forms may be represented by

$$K_iX^i,$$

where $K_i=(0, 0, 0, 0, 1)$;

[7] J. Thomae, *Ueber orthogonale Invarianten der Curven dritter Ordnung*, Berichte der Saechsischen Akademie der Wissenschaften, Leipzig, vol. 51 (1899), pp. 317–353.

[8] R. Weitzenböck, *Die Invarianten der Galilei-Newton-Gruppe*, Mathematische Annalen, vol. 80 (1920).

$$L_iX^i,$$

where $L_i = (0, 0, 0, 1, 0)$;

$$Q^{ij}U_iU_j,$$

where

$$Q^{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here $X^iU_i = 0$.

From this approach we are led to the following theorem:

THEOREM 4. *Every concomitant in Galilei-Newton geometry for a set of ground forms in X^i and dual variables U_i is expressible by composition as a tensor of order zero with the coefficient tensors of the ground forms and the tensors*

$$\epsilon^{i_1\cdots i_5},\ \epsilon_{i_1\cdots i_5},\ K_i,\ L_i,\ Q^{ij},\ X^i,\ U_i.$$

This theorem, in directness of approach and in simplicity of application, seems to have advantages over the concluding statement of Weitzenböck, which concerns itself with twelve basic symbolic factors for the Galilei-Newton group.

UNIVERSITY OF ALABAMA

A NOTE ON QUASI-METRIC SPACES[1]

GEORGE E. ALBERT

1. **Introduction.** In a recent abstract[2] the author introduced an important application of a class of neighborhood spaces in which pairs of points are required to satisfy only a very weak separation axiom (due to Kolmogoroff):

(K) *If x and y are distinct points, then at least one of these points has a neighborhood which does not contain the other.*

In the present note it is shown that a very useful generalized distance function may be defined in certain of these spaces.

Clearly, any such distance function must be an asymmetric one. W. A. Wilson[3] considered the definition of asymmetric distances in certain spaces which satisfy stronger separation axioms than K. It is shown here that a slight modification of one of the axioms in [W] allows the extension of a large part of the theory developed there to spaces subject to K.

Since many of the theorems and proofs in [W] remain valid here with only very obvious changes, this note will be limited to a mere sketch concerning new properties which arise under the weaker axioms used. The reader should have no difficulty in adapting the more complete discussion given in [W] to the case studied here.

2. **The distance function.** The symbol 1 will denote a space of points; points and point sets in 1 will be denoted by small and capital letters respectively.

The space 1 will be said to be *quasi-metric* if for every pair of points x, y in 1 there are defined two non-negative numbers xy and yx, not necessarily equal, which satisfy the following postulates:

I. $xy=yx=0$ *if and only if* $x=y$;[4]

II. *for any three points,* $xy \leqq xz+zy$.

If for some pair of points $xy=0\neq yx$, then the point x will be said to be *adjacent* to the point y. It is to be emphasized that, regardless

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[1] Presented to the Society, April 13, 1940.

[2] Abstract 46-3-138, this Bulletin.

[3] W. A. Wilson, *On quasi-metric spaces*, American Journal of Mathematics, vol. 53 (1931), p. 675. Hereafter this paper will be referred to as [W].

[4] In [W] the stronger axiom: $xy=0$ if and only if $x=y$ was used. This excludes the case $xy=0\neq yx$ allowed here.

of the connotation of symmetry familiar with the word adjacent, the relationship to which it refers here is an asymmetric one.

3. **Limits, closed and open sets.** The nature of the function xy leads to the definition of three separate types of limit points and the corresponding classes of closed and open sets. A point p will be called a *u-limit*, *l-limit*, or a *c-limit* of a point set A (of a sequence $\{x_n\}$) if for every positive number ϵ there exists a point x in $A-p$ such that $px<\epsilon$, $xp<\epsilon$, or both simultaneously, respectively (if there is a number N such that $px_n<\epsilon$, and so on, for $n>N$).

The point set $S_u(a, r)$ consisting of all points x such that $ax<r$ will be called a *u-sphere* of center a and radius r. Similarly one defines the spheres $S_l(a, r)$ and $S_c(a, r)$ by the relations $xa<r$ and both $ax<r$ and $xa<r$.

A point set will be termed *u-open*, *l-open*, or *c-open* if for every point p in the set some sphere S_u, S_l, or S_c, respectively, about p is contained entirely within the set. Such sets will be denoted by G_u, G_l, and G_c respectively. A point set will be termed *u-closed*, *l-closed*, or *c-closed* if it contains all of its limit points of the respective class. These sets will be denoted by F_u, F_l, and F_c respectively.

The space 1 will be called *separable* if there exists a denumerable sequence of points which is everywhere dense in 1 in the sense of c-limits.

Just as in the theory of ordinary topological spaces, one proves that the complement of a set G_u (G_l or G_c) is a set F_u (F_l or F_c). It is evident that every set F_u or F_l is also a set F_c and that every set G_u or G_l is also a set G_c; however the converses are not true.

It is to be noted that a point p may be a u-limit and at the same time an l-limit of a point set A without being a c-limit of that set. On the other hand, for sequences one has the following theorem:

THEOREM 1. *If $x=u$-lim x_n and $x=l$-lim x_n then $x=c$-lim x_n. Moreover, there exists no other c-limit of the sequence $\{x_n\}$ and the Cauchy criterion of convergence is satisfied by the sequence.*

PROOF. The first clause is immediate from the definitions.

Suppose that x and y are both c-limits of the sequence $\{x_n\}$. It follows that for an arbitrary positive number ϵ there exist numbers n' and n'' such that $xy\leqq xx_n+x_ny<\epsilon$ for $n>n'$ and $yx\leqq yx_n+x_nx<\epsilon$ for $n>n''$. Thus $xy=0=yx$ and by I, $x=y$.

Again, $x=c$-lim x_n implies that for arbitrary $\epsilon>0$ there is an n' such that $x_nx_m\leqq x_nx+xx_m<\epsilon$ for $n>n'$ and the Cauchy criterion is satisfied.

The following theorems indicate the relations between the several limits of a sequence and the property of adjacency.

THEOREM 2. *If $x=l\text{-}\lim x_n$ and $y=u\text{-}\lim x_n$ for the same sequence $\{x_n\}$, then either $x=y$ or else y is adjacent to x.*

The proof is immediate from $yx \leqq yx_n+x_nx$ and the fact that $yx_n \to 0$ and $x_nx \to 0$.

COROLLARY. *If $x=c\text{-}\lim x_n$ and $y=u\text{-}\lim x_n$ then either $y=x$ or y is adjacent to x; if $y=l\text{-}\lim x_n$, then either $y=x$ or x is adjacent to y.*

It is worthy of note that if a point x is adjacent to a point y then x is a u-limit of the point set consisting of the single point y. Thus, if the usual definition of connected set be adopted with limits taken in the u-sense, the point set $(x+y)$ forms a connected set.

4. **Topological versus quasi-metric spaces.** Hausdorff[5] stated the following axioms for topological spaces:

(A) Every point p has a neighborhood U_p. For every U_p, $p \in U_p$.

(B) For every two neighborhoods U_p and V_p of the point p, there is a third $W_p \subset U_p V_p$.

(C) For every point q in U_p, there is a neighborhood $U_q \subset U_p$.

(D) To each point p there corresponds a denumerable set of neighborhoods such that every neighborhood U_p contains a neighborhood of that set.

(E) There exists a denumerable set of neighborhoods $\{U_n\}$ such that if U_p is any neighborhood of the point p, then there is an n such that $p \in U_n \subset U_p$.

THEOREM 3. *Let* 1 *denote a quasi-metric space and, for each point x in* 1, *let the class of spheres $S_u(x, r)$ of rational radii r be termed neighborhoods of x. Then* 1 *is a topological space satisfying the axioms* A, B, C, D, *and* K. *The u-limits of sequences of points in* 1 *are the topological limits of the sequences and conversely. If the space is separable, Axiom* D *may be replaced by the stronger Axiom* E. *The theorem remains valid if the spheres S_u are replaced by the spheres S_l and u-limits by l-limits.*

The proof is evident.

THEOREM 4. *Let* 1 *be a neighborhood space satisfying the axioms* A, B, C, E, *and* K. *Then distances can be defined in* 1 *so that the space is a separable, quasi-metric space. A topological limit of a sequence of*

[5] Hausdorff, *Mengenlehre*, Berlin, 1935, pp. 228, 229.

points in 1 *is a u-limit of the sequence and conversely. The theorem remains valid if u-limit is replaced by l-limit provided that a corresponding change be made in the distance function.*

Since the proof of this theorem is long but follows that of Theorem IV, §7, in [W] with only a few minor changes, it seems sufficient to state the definition of the distance function for the case of u-limits.

Let the denumerable system of neighborhoods postulated by Axiom E be denoted by $\{U_n\}$. For any pair of points x and y in 1 and for each $n=1, 2, 3, \cdots$, define the function $f_n(x, y)$ to be equal 1 if x is in U_n and y is in $1-U_n$, or equal to 0 in case x is in $1-U_n$ or in case $x+y$ is in U_n. Then define the function

$$xy = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x, y).$$

It should be remarked that a quasi-metric is not necessarily a continuous function of either variable.

5. **Applications.** The distance function described above furnishes an exceedingly simple topologization for a very general type of collection of closed point sets. Let S denote a compact metric space and let $S=\sum X$ be any decomposition of S into closed point sets. It is not assumed that the sets of the decomposition are disjoint. However it is assumed that if X_1 and X_2 are distinct elements of the decomposition which overlap, then $X_1X_2=X$ is again an element of the decomposition. Moreover, if X_3 is any element which intersects X, then $X_1X_3=X=X_2X_3$. Such a collection is afforded by the *upper semi-continuous collection of the second type* introduced by R. L. Moore[6] and, in particular, by the so-called hyperspace of cyclic elements of a continuous curve. If, now, the elements of the collection are thought of as points of a new space 1, the topologization is introduced as follows: if X and Y are two elements of 1, the distance function $XY=$ the greatest lower bound of the numbers r such that, as seen in the space S, every point x of the set X has a distance from the set Y which is less than r. The space is easily shown to be quasi-metric in the sense described in the preceding paragraphs.

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[6] The Rice Institute Pamphlet, vol. 23, no. 1, 1936, p. 42.

CESÀRO SUMMABILITY OF A CLASS OF SERIES

M. S. MACPHAIL

The following theorem has recently been proved by H. L. Garabedian:[1] If a_n is a polynomial in n of degree $k-1$, the series $\sum_{n=0}^{\infty}(-1)^n a_n$ is summable (C, k) but not $(C, k-1)$ to the value $\sum_{i=0}^{k-1}2^{-i-1}\Delta^i a_0$. Here and elsewhere in this paper k is understood to be a fixed positive integer.

Our present object is to obtain some extensions of this result. One of these may be stated at once, the proof being given at the end of the paper.

THEOREM 2. *Let a_n be a polynomial in n of degree $k-1$, and let z be a complex number such that $|z|=1$, $z\neq 1$. Then the series $\sum_{n=0}^{\infty}a_n z^n$ is summable (C, k) but not $(C, k-1)$ to the value $-\sum_{m=0}^{k-1}z^m(z-1)^{-m-1}\Delta^m a_0$.*

Before stating our first theorem we require the following definitions. Let f_n be a periodic function of the integer n; that is to say, let there be an integer p such that $f_{n+p}=f_n$ for all values of n. Let $M(f)$ denote the mean value of f_n over a period, thus $M(f)=(1/p)\sum_{n=0}^{p-1}f_n$. Suppose now that f_n is a periodic function of mean value zero; that is, let $M(f)=0$. Writing $f(n)$, $a(n)$ in place of f_n, a_n, set

$$
\text{(a)}\qquad
\begin{aligned}
f_1(n) &= \sum_{j=0}^{n} f(j), & c_1 &= M(f_1), & F_1(n) &= f_1(n) - c_1,\\
f_2(n) &= \sum_{j=0}^{n} F_1(j), & c_2 &= M(f_2), & F_2(n) &= f_2(n) - c_2,\\
f_3(n) &= \sum_{j=0}^{n} F_2(j), & c_3 &= M(f_3), & F_3(n) &= f_3(n) - c_3,
\end{aligned}
$$

and so on; this procedure ensures that $F_i(n)$ is periodic with mean value zero $(i=1, 2, \cdots)$. In terms of these definitions we prove the following theorem:

THEOREM 1. *Let $a(n)$ be a polynomial in n of degree $k-1$, and $f(n)$ a periodic function with mean value zero. Then the series $\sum_{n=0}^{\infty}f(n)a(n)$ is summable (C, k) but not $(C, k-1)$ to the value $\sum_{i=0}^{k-1}c_{i+1}\Delta^i a_0$.*

If $f(n)=(-1)^n$ it is easily verified that $c_{i+1}=2^{-i-1}$, which shows

[1] This Bulletin, vol. 45 (1939), pp. 592–596. In the theorem as here stated it is required that a_n satisfy the conditions $\Delta^{k-1}a_0\neq 0$, $\Delta^i a_0=0$ $(i\geqq k)$. It is easily seen that this is equivalent to requiring a_n to be a polynomial in n of degree $k-1$.

that the present theorem includes that of Garabedian as a special case. The method of proof is entirely different.

We shall make use of the notation of finite differences. We recall that $\Delta a(n)=a(n)-a(n+1)$, $\Delta^2 a(n)=\Delta(\Delta a(n))$, $\cdots$, and $E^j a(n)=a(n+j)$. We shall require the well known equation, in symbolic form,

$$\Delta^k = (1-E)^k = 1 - C_{k,1}E + C_{k,2}E^2 - \cdots + (-1)^k E^k, \tag{1}$$

where the $C_{k,i}$ are the binomial coefficients. We also write $n^{(k)}=n(n-1)(n-2)\cdots(n-k+1)$, and by convention $n^{(0)}=1$, $\Delta^0 a(n)=a(n)$; then the successive differences of $n^{(k)}$ are given by

$$\Delta^i n^{(k)} = (-1)^i k^{(i)} n^{(k-i)}, \qquad 0 \leqq i \leqq k. \tag{2}$$

We require two algebraic identities, which are stated as lemmas.

LEMMA 1. *We have, for r a positive integer,*

$$\sum_{n=0}^{r} f(n)a(n) = \sum_{i=0}^{k-1} c_{i+1}\Delta^i a(0) + \sum_{i=0}^{k-1} F_{i+1}(r)\Delta^i a(r+1) + \sum_{n=0}^{r} F_k(n)\Delta^k a(n).$$

For,

$$\begin{aligned}
\sum_{n=0}^{r} f(n)a(n) &= f(0)a(0) + f(1)a(1) + \cdots + f(r)a(r) \\
&= f_1(0)a(0) + [f_1(1) - f_1(0)]a(1) + \cdots \\
&\quad + [f_1(r) - f_1(r-1)]a(r) \\
&= [F_1(0) + c_1]a(0) + [F_1(1) - F_1(0)a](1) + \cdots \\
&\quad + [F_1(r) - F_1(r-1)]a(r) \\
&= c_1 a(0) + F_1(0)[a(0) - a(1)] + \cdots \\
&\quad + F_1(r-1)[a(r-1) - a(r)] \\
&\quad + F_1(r)[a(r) - a(r+1)] + F_1(r)a(r+1) \\
&= c_1 a(0) + F_1(r)a(r+1) + \sum_{n=0}^{r} F_1(n)\Delta a(n).
\end{aligned}$$

The lemma follows by repeated applications of this device.

LEMMA 2. *The expressions*

$$\frac{(r+1-n)(r+2-n)\cdots(r+k-n)}{(r+1)(r+2)\cdots(r+k)} \tag{3}$$

and

$$1 - C_{k,1}\frac{n}{r+1} + C_{k,2}\frac{n(n+1)}{(r+1)(r+2)} - C_{k,3}\frac{n(n+1)(n+2)}{(r+1)(r+2)(r+3)} + \cdots + (-1)^k \frac{n(n+1)\cdots(n+k-1)}{(r+1)(r+2)\cdots(r+k)} \tag{4}$$

are algebraically identical.

To prove this, set

$$G(t) = \frac{n(n+1)\cdots(n+t-1)}{(r+1)(r+2)\cdots(r+t)}, \qquad t \text{ a positive integer},$$

$$G(0) = 1.$$

Then, if $EG(t) = G(t+1)$, the expression (4) may be written

$$G(0) - C_{k,1}EG(0) + C_{k,2}E^2G(0) - \cdots + (-1)^k E^k G(0) = (1-E)^k G(0) = \Delta^k G(0),$$

by (1). We have therefore to show that $\Delta^k G(0)$ is equal to the expression (3). We write

$$\Delta G(t) = \frac{n(n+1)\cdots(n+t-1)}{(r+1)(r+2)\cdots(r+t)} - \frac{n(n+1)\cdots(n+t)}{(r+1)(r+2)\cdots(r+t+1)} = \frac{n(n+1)\cdots(n+t-1)}{(r+1)(r+2)\cdots(r+t+1)}(r+1-n), \qquad t \geqq 1.$$

In particular

$$\Delta G(1) = \frac{n}{(r+1)(r+2)} - (r+1-n).$$

Also, since $G(0)=1$ and since $G(1)=n/(r+1)$, we have $\Delta G(0) = (r+1-n)/(r+1)$. Similarly we find

$$\Delta^2 G(t) = \frac{n(n+1)\cdots(n+t-1)}{(r+1)(r+2)\cdots(r+t+2)}(r+1-n)(r+2-n),$$

$$\Delta^2 G(1) = \frac{n}{(r+1)(r+2)(r+3)}, \quad \Delta^2 G(0) = \frac{(r+1-n)(r+2-n)}{(r+1)(r+2)}.$$

Proceeding in this way, we obtain the desired result by induction.

PROOF OF THEOREM 1. The series $\sum_{n=0}^{\infty} f(n)a(n)$ is said to converge (C, k) to the value S provided the expression

$$\sum_{n=0}^{r} \frac{(r+1-n)(r+2-n)\cdots(r+k-n)}{(r+1)(r+2)\cdots(r+k)} f(n)a(n)$$

tends to S as r becomes infinite. Now any polynomial $a(n)$ of degree $k-1$ can be expressed in the form $A_1(n-1)^{(k-1)}+A_2(n-1)^{(k-2)}+\cdots$ where $A_1, A_2, \cdots$ are constants. Hence we need only consider

$$\sum_{n=0}^{r} \frac{(r+1-n)(r+2-n)\cdots(r+k-n)}{(r+1)(r+2)\cdots(r+k)} (n-1)^{(k-1)} f(n),$$

which by Lemma 2 is equal to

$$(5) \qquad \sum_{n=0}^{r} \left\{1 - C_{k,1}\frac{n}{r+1} + C_{k,2}\frac{n(n+1)}{(r+1)(r+2)} - \cdots\right\} (n-1)^{(k-1)} f(n).$$

We consider the terms of this expression separately. By (2) and Lemma 1 we have

$$\sum_{n=0}^{r} (n-1)^{(k-1)} f(n) = \sum_{i=0}^{k-1} c_{i+1}(-1)^i (k-1)^{(i)} (-1)^{(k-1-i)} + \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i (k-1)^{(i)} r^{(k-1-i)},$$

$$\sum_{n=0}^{r} n^{(k)} f(n) = \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i k^{(i)} (r+1)^{(k-i)} + (-1)^k k! \sum_{n=0}^{r} F_k(n),$$

$$\sum_{n=0}^{r} (n+1)^{(k+1)} f(n) = \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i (k+1)^{(i)} (r+2)^{(k+1-i)} + (-1)^k (k+1)^{(k)} \sum_{n=0}^{r} (n+1) F_k(n),$$

. . . . . ,

$$\sum_{n=0}^{r} (n+k-1)^{(2k-1)} f(n) = \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i (2k-1)^{(i)} (r+k)^{(2k-1-i)} + (-1)^k (2k-1)^{(k)} \sum_{n=0}^{r} (n+k-1)^{(k-1)} F_k(n).$$

Multiply the above equations by

$$1, \quad -\frac{C_{k,1}}{r+1}, \quad \frac{C_{k,2}}{(r+1)(r+2)}, \quad \cdots,$$

respectively, and add. We then have on the left the expression (5) and on the right

$$\sum_{i=0}^{k-1} c_{i+1}(-1)^i(k-1)^{(i)}(-1)^{(k-1-i)} + \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i r^{(k-1-i)}(1-E)^k(k-1)^{(i)} + \text{terms involving } \sum_{n=0}^{r}.$$

Now, by (1), $(1-E)^k(k-1)^{(i)} = \Delta^k(k-1)^{(i)} = 0$ for $0 \leqq i \leqq k-1$. Also, the terms involving $\sum_{n=0}^{r}$ can easily be shown to approach zero as r becomes infinite. We have therefore proved that the series $\sum_{n=0}^{\infty}(n-1)^{(k-1)}f(n)$ converges (C, k) to the value $\sum_{i=0}^{k-1}c_{i+1}(-1)^i \cdot (k-1)^{(i)}(-1)^{(k-1-i)}$, or, setting $b(n) = (n-1)^{(k-1)}$, $\sum_{n=0}^{\infty}b(n)f(n)$ converges (C, k) to the value $\sum_{i=0}^{k-1}c_{i+1}\Delta^i b(0)$.

Theorem 1 now follows without difficulty. It is easily seen on examining the argument that no order of summation less than k will serve.

We now prove Theorem 2. If am z is a rational multiple of π, say am $z = 2\pi q/p$, where p and q are integers, the function $f(n) = z^n$ is periodic with period p and mean value zero. It is easily verified that, for $m \geqq 0$, the equations (a) become

(b)
$$f_{m+1}(n) = \frac{z^m}{(z-1)^m}\frac{z^{n+1}-1}{z-1}, \qquad c_{m+1} = -\frac{z^m}{(z-1)^{m+1}},$$
$$F_{m+1}(n) = \frac{z^{m+1}}{(z-1)^{m+1}}z^n,$$

whence Theorem 2 follows at once, as a special case of Theorem 1. If am z is not a rational multiple of π, then we define $M(f)$ as $\lim_{r\to\infty}(1/r)\sum_{n=0}^{r-1}f(n)$. Then the equations (b) still hold, and the theorem follows by the same argument as Theorem 1.

ACADIA UNIVERSITY

A CHARACTERIZATION OF THE GROUP OF HOMOGRAPHIC TRANSFORMATIONS

P. O. BELL

1. **Introduction.** The objectives of this note are three-fold: (1) to present a new differential geometric characterization of the group of homographic transformations of a complex variable, (2) to interpret in geometrical language the significance of the invariance of the Schwarzian derivative under a homographic transformation, and (3) to characterize a general homographic transformation by its unique association with two families of concentric circles.

2. **Preliminaries.** Let the equation

$$w = w(z) \tag{1}$$

denote a conformal representation of the points $z=x+iy$ of a region R of the z-plane on the points $w=u+iv$ of a region $\overline{R}$ of the w-plane, whereby a general curve C is transformed into a curve $\overline{C}$. Let γ and $\bar{\gamma}$ denote the curvatures of C and $\overline{C}$ at corresponding points z and w, and let s and $\bar{s}$ denote corresponding lengths of arc of C and $\overline{C}$. For a given transformation (1) it is well known that the rate of variation $ds/d\bar{s}$ is a function $\lambda(x, y)$ which may be expressed in any one of the following forms $(u_x^2+u_y^2)^{-1/2}$, $(u_x^2+v_x^2)^{-1/2}$, $(u_y^2+v_y^2)^{-1/2}$, $(v_x^2+v_y^2)^{-1/2}$.

Comenetz[1] (using a different notation) has obtained, by elementary methods, the formula

$$\bar{\gamma} = \gamma\lambda + \lambda_y \cos\theta - \lambda_x \sin\theta, \tag{2}$$

wherein $\theta=\arctan(dy/dx)$, which is *the law of transformation of curvature in conformal mapping*, and the formula

$$d\bar{\gamma}/d\bar{s} = \lambda^2 d\gamma/ds + \lambda[\lambda_{xy}\cos 2\theta + \tfrac{1}{2}(\lambda_{yy} - \lambda_{xx})\sin 2\theta], \tag{3}$$

which is *the law of transformation of the rate of change of curvature with respect to arc length, under conformal mapping.*

3. **A new characterization of the group of homographic transformations.** It is known that the most general directly conformal transformation which carries circles into circles (including straight lines) is a homographic transformation. The transformation (1) will be such a transformation if and only if

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[1] Comenetz, *Kasner's invariant and trihornometry*, The American Mathematical Monthly, vol. 45 (1938), pp. 82–87.

$$\lambda_{xx} \equiv \lambda_{yy}, \qquad \lambda_{xy} \equiv 0. \tag{4}$$

For if these relations hold, it follows from (3) that

$$d\bar{\gamma}/d\bar{s} = \lambda^2 d\gamma/ds.$$

Hence circles are transformed into circles. Conversely, if circles are transformed by (1) into circles, we must have at corresponding points of an arbitrarily selected circle C and its correspondent $\overline{C}$

$$d\bar{\gamma}/d\bar{s} = \lambda^2 d\gamma/ds = 0.$$

Equation (3) also holds. Hence, no matter what the direction of C at z may be, the following equation must hold:

$$\lambda(\lambda_{yy} - \lambda_{xx})(\sin 2\theta)/2 + \lambda\lambda_{xy} \cos 2\theta = 0. \tag{5}$$

This equation must, therefore, be satisfied independently of θ, and conditions (4) necessarily follow. We may state, therefore, the following theorem.

THEOREM 1. *A necessary and sufficient condition that a conformal transformation be a homographic transformation is that the associated function* $\lambda(x, y)$ *satisfy both of the following identities*

$$\lambda_{xx} - \lambda_{yy} \equiv 0, \qquad \lambda_{xy} \equiv 0.$$

A geometric interpretation of this condition is that at a general point w *of the curve* $\overline{C}$ *the rate of variation* $d\bar{\gamma}/d\bar{s}$ *of the curvature of* $\overline{C}$ *per unit length of arc* $\bar{s}$ *is independent of the direction of the curve* C *at the corresponding point* z.

Equation (3) shows that this is equivalent to the following characterization.

THEOREM 2. *The group of homographic transformations consists of all of the conformal transformations under which the differential form* $d\gamma ds$ *is absolutely invariant.*

The following theorem may be deduced, similarly, in consideration of equation (2).

THEOREM 3. *A necessary and sufficient condition that a transformation* (1) *be a directly conformal collineation is that the associated function* $\lambda(x, y)$ *satisfy both of the identities*

$$\lambda_x \equiv 0, \qquad \lambda_y \equiv 0.$$

Under this condition the differential form γds *is an absolute invariant of the transformation* (1).

Since $\gamma = d\theta/ds$ where $dz/ds = e^{i\theta}$, the differential form $d\gamma ds$ may be written in the form

$$(d^2\theta/ds^2)(ds)^2 \tag{6}$$

wherein $\theta = -i \log (dz/ds)$.

4. **The Schwarzian derivative.** Consider a curve defined by

$$z = x(t) + iy(t),$$

wherein x and y are functions of a real variable t. It is known that the Schwarzian derivative

$$\{z, t\} \equiv (d^3z/dt^3)/(dz/dt) - \tfrac{3}{2}[(d^2z/dt^2)/(dz/dt)]^2$$

is an absolute invariant under the homographic transformations. Let us investigate the geometric significance of the invariance of this derivative.

We find that

$$z''/z' = i\gamma \tag{7}$$

where accents indicate differentiation with respect to s. On differentiating the members of equation (7) with respect to s we obtain

$$z'''/z' - (z''/z')^2 \equiv id\gamma/ds. \tag{8}$$

Making use of (7) and (8) we deduce

$$\{z, s\} \equiv id\gamma/ds + \gamma^2/2. \tag{9}$$

If we make a change of variable by the formula[2]

$$\{z, t\} \equiv \{z, s\}(ds/dt)^2 + \{s, t\}, \tag{10}$$

we obtain

$$\{z, t\} \equiv (id\gamma/ds + \gamma^2/2)(ds/dt)^2 + \{s, t\}. \tag{11}$$

The real and imaginary components of $\{z, t\}$,

$$R \equiv (\gamma^2/2)\,(ds/dt)^2 + \{s, t\}, \quad I \equiv (d\gamma/ds)\,(ds/dt)^2,$$

are, themselves, absolute invariants of the group of homographic transformations. Thus, if (1) is homographic, we have

$$(\gamma^2/2)(ds/dt)^2 + \{s, t\} \equiv (\bar{\gamma}^2/2)(d\bar{s}/dt)^2 + \{\bar{s}, t\}, \tag{12}$$

$$(d\gamma/ds)(ds/dt)^2 \equiv (d\bar{\gamma}/d\bar{s})(d\bar{s}/dt)^2. \tag{13}$$

[2] For the change of variable formula see, for example, Ford, *Automorphic Functions*, McGraw-Hill, 1929, p. 99.

If now we put $t=s$ in (12), we find

$$2\{\bar{s}, s\} \equiv \gamma^2 - \bar{\gamma}^2(d\bar{s}/ds)^2. \tag{14}$$

Likewise equation (12) yields

$$2\{s, \bar{s}\} \equiv \bar{\gamma}^2 - \gamma^2(ds/d\bar{s})^2 \tag{15}$$

on substituting $\bar{s}$ for t.

Equation (13) expresses the invariance of the form $d\gamma ds$. Making use of this equation and equations (14) and (15), we deduce

$$2\{\bar{s}, s\} \equiv \gamma^2 - \bar{\gamma}^2(d\gamma/d\bar{\gamma})^2, \tag{16}$$

$$2\{s, \bar{s}\} \equiv \bar{\gamma}^2 - \gamma^2(d\bar{\gamma}/d\gamma)^2. \tag{17}$$

These equations express the significance of the invariance of the Schwarzian derivatives $\{z, s\}$ and $\{\bar{z}, \bar{s}\}$ as intrinsic geometric relations between any pair of curves C, $\overline{C}$ which correspond under a homographic transformation. To complete the geometric interpretations of (16) and (17) let us recall the significance of the Schwarzian derivative of a real function.[3] Consider two real functions $\sigma=\sigma(s)$ and $\bar{\sigma}=\bar{\sigma}(s)$ which are chosen to satisfy

$$\{\sigma, s\} \equiv \{\bar{\sigma}, s\}$$

identically in s. This relation is necessary and sufficient that $\sigma(s)$ and $\bar{\sigma}(s)$ be connected by a homographic transformation

$$\bar{\sigma}(s) = [a\sigma(s) + b]/[c\sigma(s) + d], \tag{18}$$

wherein a, b, c, d are constants and $ad-bc\neq 0$. The relation (18) is also necessary and sufficient that corresponding to any set of four values $s=s_j$, $(j=1, 2, 3, 4)$, the cross-ratios

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \qquad (\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3, \bar{\sigma}_4)$$

are identical. If as s varies, points P and $\overline{P}$ describe curves whose corresponding lengths of arc are defined by $\sigma=\sigma(s)$ and $\bar{\sigma}=\bar{\sigma}(s)$, the movements of these points will be called *projectively applicable*. This designation is suggested by the property that the development of these movements along a straight line produces projectively equivalent rectilinear movements.

Corresponding to a real single-valued differentiable function $\sigma=\sigma(s)$ there exists a *class* $\mathfrak{S}_{\sigma(s)}$ *of projectively applicable movements* to which a movement defined by $\bar{\sigma}=\bar{\sigma}(s)$ will be said to belong if $\{\bar{\sigma}, s\} \equiv \{\sigma, s\}$. We shall call the Schwarzian $\{\sigma, s\}$ *the absolute projective acceleration*

[3] Cartan, *Leçons sur la Théorie des Espaces*, Paris, Gauthier-Villars, 1937, p. 3.

of the movements of the class $\mathfrak{S}_{\sigma(s)}$ or simply the *absolute projective acceleration*[4] $\{\sigma, s\}$.

We may now state the following theorem:

THEOREM 4. *The invariance of the Schwarzian derivative* $\{z, t\}$ *yields the identities* (16) *and* (17) *which express the absolute projective accelerations* $\{\bar{s}, s\}$ *and* $\{s, \bar{s}\}$ *algebraically in terms of the squares of the curvatures* γ *and* $\bar{\gamma}$ *and the square of their rate of variation* $d\gamma/d\bar{\gamma}$.

Other interesting identities may be obtained by forming various combinations of (13), (14) and (15). One of these has the surprisingly simple form

$$\bar{\gamma}^2/\{s, \bar{s}\} + \gamma^2/\{\bar{s}, s\} \equiv 2. \tag{19}$$

5. **The magnimetric circles.** Let us consider a homographic transformation

$$w = (az + b)/(cz + d), \tag{20}$$

where a, b, c, d are constants and $ad - bc = 1$. Since $\lambda(x, y) = |dz/dw|$ and for (20) dz/dw is defined by $dz/dw = (cz+d)^2$, we may write

$$\lambda(x, y) \equiv (cz + d)(\bar{c}\bar{z} + \bar{d}), \tag{21}$$

where $\bar{z}, \bar{c}, \bar{d}$ denote the conjugate imaginaries of z, c, d. Let the value of λ defined at the point $z_1 = x_1 + iy_1$ be denoted by λ_1. By making use of (20), (21) and the equation

$$z = (-dw + b)/(cw - a), \tag{22}$$

for the inverse of (20), the proof of the following theorem may be supplied by the reader.

THEOREM 5. *Through a point* z_1, *in the* z-*plane* (*excluding* $z_1 = \infty$ *and* $z_1 = -d/c$), *there passes just one circle which by* (20) *is magnified in all of its elements of arc length by the constant multiple* $1/\lambda_1$. *Similarly, through the point* w_1, *which corresponds by* (20) *to the point* z_1, *there passes just one circle which by* (22) *is magnified in all of its elements of arc-length by the constant multiple* λ_1.

These circles will be called *magnimetric circles of the transformations* (20) *and* (22). The totality of the magnimetric circles in the z-plane form a family of concentric circles, a general one of which is defined by

$$(cz + d)(\bar{c}\bar{z} + \bar{d}) = \lambda, \qquad \lambda = \text{const.} \tag{23}$$

[4] Loc. cit., Footnote 3, p. 3. Cartan, in considering rectilinear motion defined by $x = x(t)$ has called the Schwarzian $\{x, t\}$ *"l'accélération projective du mouvement."*

The corresponding magnimetric circles in the w-plane form the family of concentric circles, a general one of which is defined by

$$(cw - a)(\bar{c}\bar{w} - \bar{a}) = 1/\lambda. \tag{24}$$

Let $r(z)$ and $\rho(w)$ denote the radii of the circles (23) and (24), respectively. We have, clearly, that

$$r^2(z) = \lambda/c\bar{c}, \qquad \rho^2(w) = 1/\lambda c\bar{c}. \tag{25}$$

When $\lambda = 1$, equations (24) and (25) represent the isometric[5] circles.

We shall refer to families in the z- and w-planes as z- and w-families, respectively. By making use of equations (23) and (24) in connection with (20), the following theorem is obtained.

THEOREM 6. *There are ∞^2 transformations of the form* (20) *which transform a z-family of concentric circles with an arbitrarily selected center z_0 into a w-family of concentric circles with an arbitrarily selected center w_0. There are ∞^1 of these which transform a selected circle of the z-family into a selected circle of the w-family. On requiring this pair of circles to correspond, a one-to-one correspondence among the other members of the two families is established. Finally, there is just one transformation of this infinite system which transforms a selected point z_1 on any circle of the z-family into a selected point w_1 on the corresponding circle of the w-family. The circles of these z- and w-families are the magnimetric circles of the transformation, and the product of any corresponding pair of radii is equal to $1/|c|^2$.*

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[5] Loc. cit., Footnote 2, pp. 23–27.

ON THE MAPPING OF QUADRATIC FORMS[1]

LLOYD L. DINES

The development of this paper was suggested by a theorem proposed by Bliss, proved by Albert,[2] by Reid,[3] and generalized by Hestenes and McShane.[4] That theorem had to do with two quadratic forms $P(z)$ and $Q(z)$ in real variables $z^1, z^2, \cdots, z^n$ with real coefficients, and may be stated as follows:

If $P(z)$ is positive at each point $z \neq (0)$ at which $Q(z) = 0$, then there is a real number μ such that the quadratic form $P(z) + \mu Q(z)$ is positive definite.[5]

If one considers the set of points $\mathfrak{M}$ in the xy-plane into which the z-space is mapped by the transformation

$$x = P(z), \qquad y = Q(z), \tag{1}$$

he will note that the above theorem may be interpreted as asserting the existence of a *supporting line* of the map $\mathfrak{M}$ which has contact with $\mathfrak{M}$ only at $(x, y) = (0, 0)$. This suggests that the theorem is related to the theory of convex sets.

In the present paper it is proven (Theorem 1) that $\mathfrak{M}$ is a *convex* set. Furthermore it is proven (Theorem 2) that if $P(z)$ and $Q(z)$ have no common zero except $z = (0)$, then $\mathfrak{M}$ is *closed*, and is either the entire xy-plane or an angular sector of angle less than π. Immediate corollaries include not only the theorem quoted above, but also statements of criteria for the existence of (1) semi-definite, and (2) definite linear combinations $\lambda P(z) + \mu Q(z)$. The author hopes in a subsequent paper to obtain analogous results for the general case of m quadratic forms.

Throughout the paper it is to be understood without further statement that $P(z)$ and $Q(z)$ are quadratic forms in $z^1, z^2, \cdots, z^n$, with real coefficients, the variables z^i being restricted to real values.

1. **Convexity, and the condition for $\lambda P(z) + \mu Q(z) \geqq 0$.** We give first the following theorem.

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[1] Presented to the Society, December 31, 1940.

[2] This Bulletin, vol. 44 (1938), p. 250.

[3] This Bulletin, vol. 44 (1938), p. 437.

[4] Transactions of this Society, vol. 47 (1940), p. 501.

[5] While the present paper was in press, Professor N. H. McCoy kindly called the author's attention to the fact that this theorem was proven first by Paul Finsler: *Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen*, Commentarii Mathematici Helvetici, vol. 9 (1937), pp. 188–192. Apparently this work had been overlooked by the authors referred to above.

THEOREM 1. *Under the transformation* (1), *the map* $\mathfrak{M}$ *of the z-space onto the xy-plane is convex.*

If A is a point of the map, distinct from the origin O, every point of the ray OA belongs to the map, since $P(rz)=r^2P(z)$ and $Q(rz)=r^2Q(z)$ for every real number r. Hence, if A and B are two points collinear with O, and each belongs to $\mathfrak{M}$, then all points of the line segment AB belong to $\mathfrak{M}$.

We will therefore assume that $A(x_1, y_1)$ and $B(x_2, y_2)$ are points of $\mathfrak{M}$, not collinear with the origin, defined by

$$\begin{aligned} &x_1 = P(z_1), \qquad x_2 = P(z_2), \\ &y_1 = Q(z_1), \qquad y_2 = Q(z_2), \qquad z_i = (\overset{1}{z_i}, \overset{2}{z_i}, \cdots, \overset{n}{z_i}), \end{aligned} \tag{2}$$

and attempt to show that every point on the line segment AB belongs to $\mathfrak{M}$. Without loss of generality we will further assume that

$$x_2y_1 - x_1y_2 = k^2 > 0. \tag{3}$$

It will suffice to show analytically that if $\bar{t}$ is any *given* number such that $0<\bar{t}<1$, then the equations

$$P(z) = x_1 + \bar{t}(x_2 - x_1), \qquad Q(z) = y_1 + \bar{t}(y_2 - y_1) \tag{4}$$

admit a real simultaneous solution $z=(z^1, z^2, \cdots, z^n)$.

In (4) we make the substitution

$$z = \rho(z_1 \cos\theta + z_2 \sin\theta) \tag{5}$$

where ρ and θ are real variables, and write the results in the form

$$\begin{aligned} \rho^2 p(\cos\theta, \sin\theta) &= x_1 + \bar{t}(x_2 - x_1), \\ \rho^2 q(\cos\theta, \sin\theta) &= y_1 + \bar{t}(y_2 - y_1), \end{aligned} \tag{6}$$

where p and q are quadratic forms in $\cos\theta$, $\sin\theta$, defined by

$$\begin{aligned} p(\cos\theta, \sin\theta) &\equiv P(z_1 \cos\theta + z_2 \sin\theta), \\ q(\cos\theta, \sin\theta) &\equiv Q(z_1 \cos\theta + z_2 \sin\theta). \end{aligned} \tag{7}$$

Elimination of ρ^2 from the two equations (6) imposes upon θ the condition

$$y_1 p(\cos\theta, \sin\theta) - x_1 q(\cos\theta, \sin\theta) = \bar{t}T(\theta) \tag{8}$$

where

$$T(\theta) \equiv (y_1 - y_2)p(\cos\theta, \sin\theta) - (x_1 - x_2)q(\cos\theta, \sin\theta). \tag{9}$$

The function $T(\theta)$ is a quadratic form in $\cos\theta$, $\sin\theta$, which has the positive value k^2 at $\theta=-\pi/2$, $\theta=0$, and $\theta=\pi/2$; as can be easily veri-

fied from (7), (2), and (9). Since it can vanish for at most two values of θ between $-\pi/2$ and $\pi/2$, and must be negative between any two such values if they exist, the function $T(\theta)$ will be positive on at least one of the two intervals $-\pi/2 \leqq \theta \leqq 0$ or $0 \leqq \theta \leqq \pi/2$. We will suppose, for definiteness, that it is the latter, the argument being similar in the two cases.

We define a function $f(\theta)$ by the formula

$$f(\theta) = \frac{y_1 p(\cos\theta, \sin\theta) - x_1 q(\cos\theta, \sin\theta)}{T(\theta)}, \qquad 0 \leqq \theta \leqq \pi/2,$$

which is obviously continuous on the range indicated, and which has the further properties $f(0)=0$ and $f(\pi/2)=1$. Hence it takes on all values between 0 and 1, and in particular there is a value $\bar{\theta}$ such that $f(\bar{\theta})=\bar{t}$. This $\bar{\theta}$ is then a solution of (8).

The compatibility condition (8) being satisfied by $\theta=\bar{\theta}$, we easily satisfy the two equations (6) by taking $\rho^2=\bar{\rho}^2=k^2/T(\bar{\theta})$. And the resulting

$$z = \bar{z} = \bar{\rho}(z_1 \cos\bar{\theta} + z_2 \sin\bar{\theta})$$

given by (5) provides the required solution for (4).

COROLLARY. *A necessary and sufficient condition that there exist real* λ, μ, *such that for all real* z

$$\lambda P(z) + \mu Q(z) \geqq 0$$

is that there exist real a, b, *such that the two equations* $P(z)=a$, $Q(z)=b$ *are inconsistent for real* z.

The condition is necessary, since in its absence the map $\mathfrak{M}$ is the entire xy-plane, and every line $\lambda x+\mu y=0$ *separates* the plane into a positive half-plane and a negative half-plane, each of which contains points determined by $x=P(z)$, $y=Q(z)$.

However, if the point (a, b) does not belong to the map, no point on the ray from the origin to (a, b) belongs to the map. Hence the origin is a boundary point of the convex set $\mathfrak{M}$, and through this boundary point there passes a *supporting line* $\lambda x+\mu y=0$, such that $\lambda P(z)+\mu Q(z)\geqq 0$ for all real z.

2. **Closure, and the conditions for** $\lambda P(z)+\mu Q(z)>0$. We now prove the following theorem.

THEOREM 2. *If* $P(z)$ *and* $Q(z)$ *have no common zero except* $z=(0)$, *then* $\mathfrak{M}$ *is closed as well as convex, and is either the entire* xy*-plane or an angular sector of angle less than* π.

Since $\mathfrak{M}$ is convex, if it is not the entire xy-plane it lies entirely in some half-plane

$$ax + by \geqq 0, \qquad a^2 + b^2 = 1. \tag{10}$$

We first show that, under the stated hypothesis, $\mathfrak{M}$ cannot contain both rays of the boundary line $ax+by=0$. Suppose it did contain the two symmetrical points $A(b, -a)$, $B(-b, a)$, and more explicitly that

$$P(z_1) = b, \quad Q(z_1) = -a, \quad P(z_2) = -b, \quad Q(z_2) = a.$$

Since either a or b is certainly different from zero, we may assume the notation so chosen that $a>0$. Then $Q(z_1)<0$ and $Q(z_2)>0$. Hence[6] there are, in the hyperplane defined by $z=z_1u+z_2v$, two linearly independent points $z_0=z_1u_0+z_2v_0$, $z_0' =z_1u_0' +z_2v_0'$, such that

$$Q(z_0) = Q(z_0') = 0. \tag{11}$$

Consider now the quadratic form

$$\phi(u, v) = aP(z_1u + z_2v) + bQ(z_1u + z_2v)$$

in the two real variables u, v. It is easily verified that ϕ vanishes at $(u, v)=(1, 0)$ and at $(u, v)=(0, 1)$. These, together with the dependent points $(c, 0)$ and $(0, c)$, are its only possible zeros unless it vanishes identically. It does not vanish identically, since it does not vanish at (u_0, v_0) or (u_0', v_0') in view of (11) and our hypothesis. Hence, by (10), $\phi(u, v)>0$ except at $(c, 0)$ and $(0, c)$. This is clearly impossible, and the contradiction proves that the map $\mathfrak{M}$ cannot contain both points $A(b, -a)$ and $B(-b, a)$.

We now let $X(x, y)$ denote any point of $\mathfrak{M}$, and consider the angle AOX, where $A\equiv A(b, -a)$ and $O\equiv O(0, 0)$. Then $\cos AOX =(bx-ay)/(x^2+y^2)^{1/2}$. And as the point z varies over the unit hypersphere $\|z\| =1$, $\cos AOX$ is represented by the function

$$\psi(z) = \frac{bP(z) - aQ(z)}{[P^2(z) + Q^2(z)]^{1/2}}, \qquad \|z\| = 1.$$

In view of the hypothesis, $\psi(z)$ is continuous on this hypersphere; and since its values are bounded below by -1 and above by $+1$, it attains a minimum value $m\geqq -1$ and a maximum value $M\leqq 1$. It is impossible that $m=-1$ *and* $M=1$, since then the map $\mathfrak{M}$ would contain both points $A(b, -a)$ and $B(-b, a)$. Hence $\mathfrak{M}$ consists of a closed

[6] Reference may be made to Bôcher, *Introduction to Higher Algebra*, p. 151, Theorem 2.

sector bounded by rays OA' and OB' such that $\cos AOA' = M$ and $\cos AOB' = m$. And *angle* $A'OB' <$ *angle* $AOB = \pi$.

COROLLARY 1. *Necessary and sufficient conditions that there exist real* λ, μ, *such that for all real* $z \neq (0)$

$$\lambda P(z) + \mu Q(z) > 0 \tag{12}$$

are that: (1) *there exist real* a, b, *such that the two equations* $P(z) = a$, $Q(z) = b$ *are inconsistent for real* z; *and* (2) $P(z)$ *and* $Q(z)$ *have no common zero except* $z = (0)$.

The necessity is obvious. The sufficiency follows from Theorem 2. For if $(\lambda, \mu) \neq (0, 0)$ is a point of $\mathfrak{M}$ on the bisector of its angular sector, then (12) is satisfied.

COROLLARY 2. (Bliss-Albert theorem.) *If, whenever* $Q(z) = 0$ *and* $z \neq (0)$, $P(z) > 0$; *then there exists a real number* μ *such that* $P(z) + \mu Q(z)$ *is positive definite.*

The conditions of Corollary 1 are obviously satisfied with $(a, b) = (-1, 0)$. Hence there exist λ, μ, satisfying (12). If $Q(z)$ actually vanishes for some $z \neq (0)$, λ is necessarily positive and hence may be taken equal to 1.

If, on the contrary, $Q(z)$ is definite, then the map $\mathfrak{M}$ is a *closed* sector of which only the vertex (0, 0) is on the x-axis. Hence there is a line $x + \mu y = 0$ such that $x + \mu y > 0$ for all points of $\mathfrak{M}$ except (0, 0). Then $P(z) + \mu Q(z)$ is positive definite.

It is perhaps worthy of note that the two conditions of Corollary 1 are completely independent. This is shown by the following four examples.

Example 1, in which both (1) and (2) are satisfied:

$$P(u, v) \equiv u^2, \qquad Q(u, v) \equiv v^2.$$

Example 2, in which (1) is satisfied but (2) is not:

$$P(u, v) \equiv u^2, \qquad Q(u, v) \equiv uv.$$

Example 3, in which (1) is not satisfied but (2) is:

$$P(u, v) \equiv u^2 + 2uv, \qquad Q(u, v) \equiv 2uv + v^2.$$

Example 4, in which neither (1) nor (2) is satisfied:

$$P(u, v, w, t) \equiv u^2 + 2uv + w^2, \qquad Q(u, v, w, t) \equiv 2uv + v^2 + wt.$$

CARNEGIE INSTITUTE OF TECHNOLOGY

ON THE REPRESENTATIONS, $N_3(n^2)$ [1]

C. D. OLDS

1. Introduction. Let the symbol $N_r(n)$ denote the number of representations of the positive integer n in the form $n=x_1^2+x_2^2+\cdots+x_r^2$, where $x_1, x_2, \cdots, x_r$ are positive or negative integers or zero. We will agree to count the two representations $n=x_1^2+x_2^2+\cdots+x_r^2$, $n=y_1^2+y_2^2+\cdots+y_r^2$, as distinct unless simultaneously $x_\nu=y_\nu$, $\nu=1, 2, \cdots, r$. Notice that in a given representation the signs of the roots, as well as their arrangement, are relevant. A zero square, however, is supposed to have only one root.

In a letter written in 1884 to Ch. Hermite, T. J. Stieltjes[2] proved by means of elliptic functions that if $n=p^k$, $p\equiv 1 \pmod 8$, p prime, then $N_3(n^2)=6p^k$. Later in 1907, A. Hurwitz[3] stated without proof that if

$$n = 2^k m = 2^k PQ, \qquad P = \prod_{\nu=1}^{r} p_\nu^{a_\nu}, \qquad Q = \prod_{\nu=1}^{s} q_\nu^{b_\nu}, \tag{1}$$

where each p_ν is a prime$\equiv 1 \pmod 4$, and each q_ν is a prime$\equiv 3 \pmod 4$, then

$$N_3(n^2) = 6P\prod_{\nu=1}^{s}\left[q_\nu^{b_\nu} + 2\,\frac{q_\nu^{b_\nu}-1}{q_\nu-1}\right]. \tag{2}$$

This result is also implicitly contained in Stieltjes' letter mentioned above.

In 1940, G. Pall[4] showed that (2) could be derived arithmetically by an application of certain divisibility properties of the Lipschitz integral quaternions. It is the purpose of this paper to give a simple arithmetical proof of (2) by a method which has been evolved from the study of a paper by Hurwitz[5] in which he derived the analogous formula for $N_6(n^2)$.[6]

[1] This is the first part of a paper presented to the Society April 6, 1940, under the title *On the number of representations of the square of an integer as the sum of an odd number of squares.*

[2] T. J. Stieltjes, "Lettre 45," *Correspondence d'Hermite et de Stieltjes*, vol. 1, Paris, 1905, pp. 89–94.

[3] A. Hurwitz, *Mathematische Werke*, vol. 2, Basel, 1933, p. 751.

[4] G. Pall, Transactions of this Society, vol. 47 (1940), pp. 487–500. See also G. Pall, Journal of the London Mathematical Society, vol. 5 (1930), pp. 102–105. In this paper Pall gives analytical proofs of the formula for $N_r(cn^2)$, $r=3, 5, 7, 11$, c an integer.

[5] A. Hurwitz, Comptes Rendus de l'Académie des Sciences, Paris, vol. 98 (1884), pp. 504–507; *Mathematische Werke*, vol. 2, pp. 5–7. Notice that Hurwitz makes use of certain results announced by Liouville and some formulas of Stieltjes.

[6] The author wishes to acknowledge the assistance rendered him by Professor J. V. Uspensky.

2. **Two lemmas.** The arithmetical derivation of (2) depends upon the following propositions:

LEMMA 1. *Let $f(n)$ be an arbitrary arithmetical function, and suppose that $f(nn')=f(n)f(n')$ for any two integers n, n', and that $f(n)\neq 0$ for all n. If*

$$F(n) = \sum_{d|n} f(d)$$

where, in particular, $f(1)=F(1)=1$, then

$$F(nn') = \sum_{d|n,n'} \mu(d)f(d)F(n/d)F(n'/d), \tag{3}$$

where $\mu(n)$ is the Möbius function,[7] and the summation extends over all divisors common to both n and n'. If we agree to set $F(x)=0$ if x is not equal to an integer, then the summation can be extended to $d=1, 2, \cdots$ ending with a number greater than n or n'.

PROOF. Set $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, $n'=p_1^{\beta_1}p_2^{\beta_2}\cdots p_r^{\beta_r}$, where $\alpha_\nu\geqq 0$, $\beta_\nu\geqq 0$, and $p_1, p_2, \cdots, p_r$ are distinct primes. Then it follows from the definition of $F(n)$ that

$$F(p_\nu^{\alpha_\nu+\beta_\nu}) = F(p_\nu^{\alpha_\nu})F(p_\nu^{\beta_\nu}) - f(p_\nu)F(p_\nu^{\alpha_\nu-1})F(p_\nu^{\beta_\nu-1}),$$

for $\nu=1, 2, \cdots, r$, and for $\alpha_\nu\geqq 0$, $\beta_\nu\geqq 0$. It is also clear that $F(p^\alpha q^\beta)=F(p^\alpha)F(q^\beta)$ provided p and q are distinct primes. Hence

$$F(nn') = \prod_{\nu=1}^{r} F(p_\nu^{\alpha_\nu+\beta_\nu}) = \prod_{\nu=1}^{r} [F(p_\nu^{\alpha_\nu})F(p_\nu^{\beta_\nu}) - f(p_\nu)F(p_\nu^{\alpha_\nu-1})F(p_\nu^{\beta_\nu-1})]$$

$$= \sum_{d=1,2,3,\cdots} \mu(d)f(d)F(n/d)F(n'/d).$$

The second lemma we need is derived as follows: Let $f(x)=f(-x)$ be an arbitrary even function defined for integral values of x; then it is possible to show by purely arithmetical reasoning that[8]

$$\sum_{(a)} [f(d'-d'') - f(d'+d'')] = \sum_{(b)} d[f(0) - f(2d)], \tag{4}$$

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[7] The Möbius function is defined as follows: $\mu(1)=1$, $\mu(n)=0$ if n has a squared factor; $\mu(p_1p_2\cdots p_r)=(-1)^r$, if all the primes $p_1, p_2, \cdots, p_r$ are different.

[8] M. J. Liouville, Journal de Mathématique, (2), vol. 3 (1858), p. 194. Although this identity was first proved arithmetically by T. Pepin, Journal de Mathématique, (4), vol. 4 (1888), p. 94, it is more convenient to refer to the exposition in Uspensky and Heaslet, *Elementary Number Theory*, New York, 1939, p. 462.

the summations extending respectively over all positive integral solutions of the equations

(a) $2n=d'\delta'+d''\delta''=s'+s''$, d', δ', d'', δ'' odd,

(b) $n=d\delta$, δ odd.

If in (4) we replace $f(x)$ by $(-1)^{x/2}f(x)$, x even, we obtain after a few simple reductions,

$$\sum_{(a)} (-1)^{(\delta'-1)/2+(\delta''-1)/2}\left[f(d'-d'')+f(d'+d'')\right] = \sum_{(b)} d\left[f(2d)-(-1)^d f(0)\right]. \tag{5}$$

Now define two arithmetical functions $\sigma_k(n)$ and $\rho_k(n)$ as follows:

$$\sigma_k(n)=\sum_{d\mid n} d^k, \qquad \sigma_0(n)=\sigma(n), \quad 1\leqq d\leqq n;$$

$$\rho_k(n)=\sum_{n=d\delta,\,\delta \text{ odd}} (-1)^{(\delta-1)/2}d^k, \quad \rho_0(n)=\rho(n),$$

where in the second function the summation extends over all positive integral solutions d, δ of the equation $n=d\delta$, where δ is odd.

On setting $f(x)=1$ in (5), and supposing that n is odd $(=m)$ we obtain the following.

LEMMA 2.

$$\sum_{2m=s'+s''} \rho(s')\rho(s'') = \sigma_1(m),$$

where the summation extends over all positive, odd integers s', s'', satisfying the equation $2m=s'+s''$.

3. **The formula for $N_3(n^2)$.** Using the definition of n given by (1), and noticing that $N_3(2^{2k}m^2)=N_3(m^2)$, we see that we need only seek an expression for the number of solutions of the equation $m^2=x^2+y^2+z^2$. Since $m^2\equiv 1 \pmod 4$, then one of the roots of this equation must be odd, while the other two must be even. Denote by R the number of solutions in which x is even. Then it is a simple matter to verify that

$$N_3(m^2)=\tfrac{3}{2}R.$$

On the other hand, R can be expressed by the sum

$$R=\sum_{v=0,\pm1,\pm2,\cdots} N_2(m^2-4v^2),$$

which is extended over all integers ν rendering the argument non-negative. Knowing this, and using the well known result that[9]

$$N_2(n) = 4\rho(n), \qquad n \geqq 1,$$

we obtain at once

$$\begin{aligned} N_3(n^2) = N_3(m^2) &= \tfrac{8}{2} \sum_{\nu=0,\pm1,\pm2,\cdots} N_2(m^2 - 4\nu^2) \\ &= \tfrac{8}{2}\cdot 4 \sum_{\nu=0,\pm1,\pm2,\cdots} \rho(m^2 - 4\nu^2) \\ &= 6 \sum_{\nu=0,\pm1,\pm2,\cdots} \rho((m-2\nu)(m+2\nu)) = 6 \sum_{2m=a+b} \rho(ab), \end{aligned}$$

where the last sum extends over all positive odd integers a, b which satisfy the equation $2m=a+b$.

The problem is now reduced to the evaluation of the expression $\sum\rho(ab)$. To this end we use Lemma 1. Define $f(n)$ as follows:

$$f(n) = 0 \text{ if } n \text{ is even},$$

$$f(n) = (-1)^{(n-1)/2} \text{ if } n \text{ is odd}.$$

Then $f(nn')=f(n)f(n')$ for any two integers n, n' as required. For odd n

$$F(n) = \sum_{d|n} f(d) = \sum_{d|n} (-1)^{(d-1)/2} = \sum_{n=d\delta,\,\delta \text{ odd}} (-1)^{(\delta-1)/2} = \rho(n).$$

Setting $n=ab$, we obtain from Lemma 1,

$$\rho(ab) = \sum_{d=1,3,5,\cdots} \mu(d)(-1)^{(d-1)/2}\rho(a/d)\rho(b/d),$$

where $\rho(x)=0$ if x is not an integer. It follows that

$$\begin{aligned} \sum_{2m=a+b} \rho(ab) &= \sum_{2m=a+b} \sum_{d=1,3,5\cdots} \mu(d)(-1)^{(d-1)/2}\rho(a/d)\rho(b/d) \\ &= \sum_{d=1,3,5,\cdots} \mu(d)(-1)^{(d-1)/2} \sum_{2m=a+b} \rho(a/d)\rho(b/d). \end{aligned}$$

Now let d be any common divisor of a and b, and set $a=\alpha d$, $b=\beta d$, α, β odd. Then, using Lemma 2, we see that

$$\sum_{2m=a+b} \rho(a/d)\rho(b/d) = \sum_{2m/d=\alpha+\beta} \rho(\alpha)\rho(\beta) = \sigma_1(m/d), \qquad m/d \text{ odd}.$$

Consequently,

[9] For an arithmetical proof of this result see, for example, Hardy and Wright, *The Theory of Numbers*, Oxford, 1938, p. 241.

$$\sum_{2m=a+b} \rho(ab) = \sum_{d|m} \mu(d)(-1)^{(d-1)/2}\sigma_1(m/d)$$

$$= \left[\sum_{d|P} \mu(d)(-1)^{(d-1)/2}\sigma_1(P/d)\right]$$

$$\cdot \left[\sum_{d|Q} \mu(d)(-1)^{(d-1)/2}\sigma_1(Q/d)\right], \qquad m = PQ.$$

Now using the properties of $\mu(n)$, and being careful of the sign of $(-1)^{(d-1)/2}$ according as $d \mid P$ or $d \mid Q$, we can easily show, if we notice that $\sigma_1(p^\alpha) = (p^{\alpha+1}-1)/(p-1)$, p prime, that

$$\sum_{d|P} \mu(d)(-1)^{(d-1)/2}\sigma_1(P/d) = \prod_{\nu=1}^{r} [\sigma_1(p_\nu^{a_\nu}) - \sigma_1(p_\nu^{a_\nu-1})]$$

$$= \prod_{\nu=1}^{r}\left[\frac{p_\nu^{a_\nu+1}-1}{p_\nu-1} - \frac{p_\nu^{a_\nu}-1}{p_\nu-1}\right]$$

$$= \prod_{\nu=1}^{r} p_\nu^{a_\nu} = P.$$

Likewise,

$$\sum_{d|Q} \mu(d)(-1)^{(d-1)/2}\sigma_1(Q/d) = \prod_{\nu=1}^{s} [\sigma_1(q_\nu^{b_\nu}) + \sigma_1(q_\nu^{b_\nu-1})]$$

$$= \prod_{\nu=1}^{s}\left[\frac{q_\nu^{b_\nu+1}-1}{q_\nu-1} + \frac{q_\nu^{b_\nu}-1}{q_\nu-1}\right].$$

Combining these two results, we obtain the required expression for $N_3(n^2)$, namely,

$$N_3(n^2) = 6P\prod_{\nu=1}^{s}\left[q_\nu^{b_\nu} + 2\,\frac{q_\nu^{b_\nu}-1}{q_\nu-1}\right].$$

STANFORD UNIVERSITY

SOME MORE UNIFORMLY CONVEX SPACES[1]

MAHLON M. DAY

Let $\{B_i, i=1, 2, \cdots\}$ be a sequence of Banach spaces, and define $B=\mathcal{P}^p\{B_i\}$ to be the space of sequences $b=\{b_i\}$ with $b_i \in B_i$ and $\|b\|=(\sum_i \|b_i\|^p)^{1/p}<\infty$, $1<p<\infty$. It is known that B, normed in this way, is also a Banach space. Boas[2] showed that if $B_i=l^p$ for all i or if $B_i=L^p$ for all i then B is uniformly convex. Since then I have shown[3] that if $B_i=l^{p_i}$ or L^{p_i}, $1<p_i<\infty$, and if the sequence $\{p_i\}$ is not bounded away from 1 and ∞ (that is, if there do not exist $1<m \leqq M<\infty$ with $m \leqq p_i \leqq M$ for all i), then B is not uniformly convex and cannot be renormed to an isomorphic uniformly convex space. One purpose of this note is to fill in the gap between these results by means of the following theorem.

THEOREM 1. *If $B_i=l^{p_i}$ or L^{p_i}, $1<p_i<\infty$, and if the sequence $\{p_i\}$ is bounded away from 1 and ∞, then $B=\mathcal{P}^p\{B_i\}$ is uniformly convex.*

Another question which is answered here is raised by Boas (loc. cit., Footnote 3). In Boas' notation $l^p(B_0)$ is the space $\mathcal{P}^p\{B_i\}$ with $B_i=B_0$ for all i. The space $L^p(B_0)$ is the space of all Bochner integrable functions[4] f on, say, $[0, 1]$ with values in B_0 and with $\|f\|=[\int_0^1 \|f(x)\|^p dx]^{1/p}<\infty$. Boas' conjecture is verified by the next theorem.

THEOREM 2. *$l^p(B_0)$ and $L^p(B_0)$ are uniformly convex if (and, obviously, only if) B_0 is.*

Both of these results follow from the remaining theorem of this note. If all B_i, $i<\infty$, are uniformly convex there exists for each ϵ, $0<\epsilon \leqq 2$, a positive number $\delta_i(\epsilon)$ such that $\|b_i\|=\|b_i'\|=1$ and $\|b_i-b_i'\|>\epsilon$ implies $\|b_i+b_i'\|<2(1-\delta_i(\epsilon))$. The sequence $\{B_i, i<\infty\}$ will be said to have a common modulus of convexity if there is one function $\delta(\epsilon)>0$ which can be used here in place of all $\delta_i(\epsilon)$. It is clear that if we define $\delta_i(\epsilon)=\frac{1}{2}\inf[2-\|b_i+b_i'\|]$, where the infimum is taken over b_i, b_i' with $\|b_i\|=\|b_i'\|=1$ and $\|b_i-b_i'\|>\epsilon$, then such a $\delta(\epsilon)$ exists if and only if $\inf_i \delta_i(\epsilon)>0$ for every $\epsilon>0$. It is clear that $\delta(\epsilon)$ may be assumed to be a non-decreasing function of ϵ.

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[1] Presented to the Society, November 23, 1940.

[2] R. P. Boas, Jr., *Some uniformly convex spaces*, this Bulletin, vol. 46 (1940), pp. 304–311.

[3] M. M. Day, *Reflexive Banach spaces not isomorphic to uniformly convex spaces*, this Bulletin, vol. 47 (1941), pp. 313–317.

[4] S. Bochner, *Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind*, Fundamenta Mathematicae, vol. 20 (1933), pp. 262–276.

THEOREM 3. *$B=\mathcal{P}^p\{B_i\}$ is uniformly convex if and only if the B_i have a common modulus of convexity.*

The space B_i is equivalent to the closed linear subspace of those points of B of the form $\{0, 0, \cdots, 0, b_i, 0, \cdots\}$. If B is uniformly convex, it follows that the function $\delta(\epsilon)$ existing in B will be suitable as a common modulus for all the B_i.

The proof of the second part of Theorem 3 falls naturally into two cases. First we assume $\|b\| = \|b'\| = 1$, $\|b-b'\| > \epsilon$ and $\|b_i\| = \|b_i'\|$ for all i. For convenience let $\|b_i\| = \beta_i$ and $\|b_i - b_i'\| = \gamma_i$. Then for any i, $\|b_i + b_i'\| < 2(1-\delta(\gamma_i/\beta_i))\beta_i$ since b_i and b_i' both lie on the sphere of radius β_i about the origin in B_i. Hence

$$\|b + b'\| = \left(\sum_{i<\infty} \|b_i + b_i'\|^p\right)^{1/p}$$

$$\leqq 2\left\{\sum_i [(1 - \delta(\gamma_i/\beta_i))\beta_i]^p\right\}^{1/p}. \tag{1}$$

Also $\|b_i - b_i'\| \leqq \|b_i\| + \|b_i'\|$ so $\gamma_i \leqq 2\beta_i$ for all i; let E be the set of those i's such that $\gamma_i/\beta_i > \epsilon/4$ and let F be the complement of E. $\beta_i \geqq 4\gamma_i/\epsilon$ if $i \in F$ so

$$1 = \left(\sum_{i<\infty} \beta_i^p\right)^{1/p} \geqq \left(\sum_{i\in F} \beta_i^p\right)^{1/p} \geqq (4/\epsilon)\left(\sum_{i\in F} \gamma_i^p\right)^{1/p};$$

that is,

$$\left(\sum_{i\in F} \gamma_i^p\right)^{1/p} \leqq \epsilon/4$$

or

$$\left(\sum_{i\in E} \gamma_i^p\right)^{1/p} = \left(\sum_{i<\infty} \gamma_i^p - \sum_{i\in F} \gamma_i^p\right)^{1/p} \geqq (\epsilon^p - (\epsilon/4)^p)^{1/p} \geqq 3\epsilon/4.$$

Hence

$$\alpha = \left(\sum_{i\in E} \beta_i^p\right)^{1/p} \geqq 3\epsilon/8.$$

Using this and (1)

$$\begin{aligned}
\|b + b'\| &\leq 2\left[(1 - \delta(\epsilon/4))^p \sum_{i\in E} \beta_i^p + \sum_{i\in E} \beta_i^p\right]^{1/p} \\
&\leqq 2[(1 - \delta(\epsilon/4))^p \alpha^p + 1 - \alpha^p]^{1/p} \\
&\leqq 2\{1 - [1 - (1 - \delta(\epsilon/4))^p]\alpha^p\}^{1/p} \\
&\leqq 2\{1 - [1 - (1 - \delta(\epsilon/4))^p](3\epsilon/8)^p\}^{1/p}.
\end{aligned}$$

The coefficient of α^p is positive and depends only on ϵ, so, if we set the right-hand side equal to $2(1-\delta_0(\epsilon))$, then $\delta_0(\epsilon)>0$ if $\epsilon>0$, thus disposing of the first case.

In the general case we suppose only that $\|b\|=\|b'\|=1$ and that $\|b+b'\|>2(1-\delta_1(\alpha))$, where $0<\alpha\leqq 2$ and $\delta_1(\alpha)$ is any modulus of convexity which can be used in the definition of uniform convexity in l^p. Then

$$2(1-\delta_1(\alpha)) \leqq \left(\sum_i \|b_i+b_i'\|^p\right)^{1/p} \leqq \left(\sum_i (\|b_i\|+\|b_i'\|)^p\right)^{1/p} \leqq 2.$$

Since the sequences $\{\|b_i\|\}$ and $\{\|b_i'\|\}$ are in l^p, we have

$$\left(\sum_i (\|b_i\|-\|b_i'\|)^p\right)^{1/p} \leqq \alpha.$$

Let $b''=\{b_i''\}$ be the point with $b_i''=b_i'\|b_i\|/\|b_i'\|$; then $\|b_i''\|=\|b_i\|$ for each i and $\|b'-b''\|=(\sum_i(\|b_i'\|-\|b_i\|)^p)^{1/p}<\alpha$; hence, by Case I, $\|b-b''\|\leqq\epsilon/2$ if $\|b+b''\|\geqq 2(1-\delta_0(\epsilon/2))$, but $\|b+b''\| \geqq \|b+b'\|-\|b''-b'\|>2(1-\delta_1(\alpha)-\alpha/2)$.

Take α so that $0<\alpha<\epsilon/2$ and $\delta_1(\alpha)+\alpha/2<\delta_0(\epsilon/2)$; then $\|b+b'\| >2(1-\delta_1(\alpha))$ implies $\|b+b''\|>2(1-\delta_0(\epsilon/2))$ which in turn implies $\|b-b''\|<\epsilon/2$, leading finally to $\|b-b'\|\leqq\|b-b''\|+\|b'-b''\|<\epsilon/2+\epsilon/2$. Let the $\delta_1(\alpha)$ defined by this process be $\delta_2(\epsilon)$; this is greater than zero and $\|b-b'\|<\epsilon$ if $\|b\|=\|b'\|=1$ and $\|b+b'\|>2(1-\delta_2(\epsilon))$. This proves Theorem 3.

To show that Theorem 1 follows from this, use Clarkson's evaluations[5] of $\delta_p(\epsilon)$ in l^p or L^p: $\delta_p(\epsilon)=1-[1-(\epsilon/2)^r]^{1/r}$ where $r=p$ if $p\geqq 2$ and $r=p/(p-1)$ if $1<p\leqq 2$. The sequence $\{p_i\}$ with $1<p_i<\infty$ is bounded away from 1 and ∞ if and only if the sequence $\{r_i\}$ associated with $\{p_i\}$ is bounded. A little elementary calculus will convince the reader that if ϵ is fixed $\delta_p(\epsilon)$ decreases as r increases; hence if $r\geqq r_i$ for all i, $\delta_r(\epsilon)\leqq\delta_{p_i}(\epsilon)$ for all i and the sequence of spaces λ^{p_i} satisfies the hypothesis of Theorem 3 for $\lambda^{p_i}=l^{p_i}$ or L^{p_i} provided only that $\{p_i\}$ is bounded away from 1 and ∞.

The first part of Theorem 2, that $l^p(B_0)$ is uniformly convex if B_0 is, is a trivial consequence of Theorem 3. The proof for $L^p(B_0)$ involves an embedding argument similar to that used by Clarkson in analogous circumstances in L^p.

Let H be the class of simple functions in $L^p(B_0)$; that is, $h\in H$ if

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[5] J. A. Clarkson, *Uniformly convex spaces*, Transactions of this Society, vol. 40 (1936), pp. 396–414.

there exist measurable sets $E_1, \cdots, E_n$, disjoint, whose sum is the unit interval, and points $h_1, \cdots, h_n \in B_0$ such that $h = \sum_{i \leqq n} h_i \phi_{E_i}$ where ϕ_{E_i} is the characteristic function of the set E_i; that is, $\phi_{E_i}(t) = 1$ if $t \in E_i$ and is otherwise 0. H is dense in $L^p(B_0)$. If $E_1, \cdots, E_n$ are fixed, set up a correspondence between all functions in H of the form $h = \sum_{i \leqq n} h_i \phi_{E_i}$ and all points $\mathfrak{h}$ in $l^p(B_0)$ with coordinates $b_i = 0$ for $i > n$ as follows: h corresponds to $\mathfrak{h} = \{b_i\}$ if $b_i = h_i \beta_i^{1/p}$ for $i \leqq n$ where $\beta_i = m(E_i)$, $b_i = 0$ otherwise. Then $\|\mathfrak{h}\| = (\sum_{i \leqq n} h_i^p \beta_i)^{1/p} = \|h\|$ so the transformation is 1-to-1 and norm preserving.

If h, h' are of the given form, if $\|h\| = \|h'\| = 1$ and $\|h - h'\| > \epsilon$, then the same relations are true for the points $\mathfrak{h}$, $\mathfrak{h}'$ in $l^p(B_0)$; so $\|h - h'\| = \|\mathfrak{h} - \mathfrak{h}'\| < 2(1 - \delta(\epsilon))$ where $\delta(\epsilon)$ is the modulus of convexity in $l^p(B_0)$. Now for any pair h, h' of functions in H there is a partition of the unit interval into sets $E_1, \cdots, E_n$ such that h, h' are represented in the proper form using these sets. It follows that the inequality given holds for any h, $h' \in H$ with $\|h\| = \|h'\| = 1$ and $\|h - h'\| > \epsilon$. Hence by continuity of the norm in $L^p(B_0)$ and by the density of H in $L^p(B_0)$, $L^p(B_0)$ is uniformly convex and the same modulus of convexity holds in it as in $l^p(B_0)$.

We may remark that the use of the interval (0, 1) and Lebesgue measure was obviously non-essential; any measure finite or infinite on any Borel field of measurable subsets of any set might be used with integration as might be expected. The essential property of the type of integration is that the simple functions in $L^p(B_0)$ shall be dense there.

It is also noteworthy that in the proof of the last part of Theorem 2 we gave the essential part of the proof of the following corollary.

COROLLARY 1. *For fixed p with $1 < p < \infty$ and fixed uniformly convex B_0 the spaces $l^p(B_0)$ and $L^p(B_0)$ (no matter what measure is used in defining the space $L^p(B_0)$) have a common modulus of convexity, that of, for example, $l^p(B_0)$.*

In fact, since the evaluation of $\delta_2(\epsilon)$ in Theorem 3 used only $\delta(\epsilon)$ in the spaces B_i and $\delta_1(\epsilon)$ in l^p, it follows for p in any interval $1 < m \leqq p \leqq M < \infty$ and for any $\{B_i\}$ or B from a family of spaces $\{B\}$ with common modulus of convexity, that $\mathcal{P}^p\{B_i\}$ and hence $l^p(B)$ and $L^p(B)$ all have a common modulus of convexity.

UNIVERSITY OF ILLINOIS

A FLAT KLEIN BOTTLE ISOMETRICALLY EMBEDDED IN EUCLIDEAN 4-SPACE

C. TOMPKINS

There is no information available as to whether any compact manifold which is equipped with an intrinsically consistent riemannian metric may be embedded in euclidean space so that this metric is induced on it by the metric of the euclidean space. The simplest manifold which seemed to be a possible counterexample is a flat Klein bottle. An example of a flat Klein bottle embedded in euclidean 4-space is given here. The manifold of this example intersects itself. The example is offered to remove the strongest contender from the list of possible counterexamples.

The embedding equations of the Klein bottle are

$$x_1 = \cos v \cos u, \qquad x_2 = \cos v \sin u,$$

$$x_3 = 2 \sin v \cos \frac{u}{2}, \qquad x_4 = 2 \sin v \sin \frac{u}{2}.$$

The following identities are clearly satisfied:

$$x_i(u, v + 2\pi) = x_i(u, v), \qquad x_i(u + 2\pi, 2\pi - v) = x_i(u, v).$$

These are just the identifications of points which by definition of the Klein bottle convert the u, v-plane into a Klein bottle. The components of the metric tensor are

$$E = 1, \qquad F = 0, \qquad G = 1 + 3 \cos^2 v,$$

and these can be transformed into the more usual form

$$\overline{E} = 1, \qquad \overline{F} = 0, \qquad \overline{G} = 1$$

by an obvious transformation involving an elliptic function.

PRINCETON UNIVERSITY

THE R_λ-CORRESPONDENT OF THE TANGENT TO AN ARBITRARY CURVE OF A NON-RULED SURFACE

P. O. BELL

In a recent paper[1] the author defined at a general point y of a non-ruled analytic surface S the tangent line which he calls the *R_λ-correspondent* of the tangent at y to a general curve C_λ of S. It was proved[2] that (i) *a curve C_λ is a curve of Darboux if and only if at each of its points the R_λ-correspondent of the tangent to C_λ coincides with this tangent*, (ii) *a curve C_λ is a curve of Segre if and only if at each of its points the tangent to C_λ and its R_λ-correspondent are conjugate tangents of S.*

The primary purpose of this note is to present the following simple construction for the R_λ-correspondent: Let Λ denote a point of C_λ distinct from y, let U, V denote, respectively, the points of intersection of the asymptotic u- and v-curves passing through y with the asymptotic v- and u-curves passing through Λ, and let W denote the point of intersection of the tangent plane to S at y with the line joining the points U, V. *If y is held fixed while Λ tends toward y along C_λ, the point W describes a curve C_w and, except when C_λ is a curve of Segre or is tangent at y to a curve of Segre, the limit of W is the point y. The tangent at y to C_w is the R_λ-correspondent of the tangent to C_λ at y.*

The validity of this construction will be proved, and in addition the following theorem will be demonstrated:

A curve C_λ is a curve of Segre if and only if for a general point y of C_λ the limit of W as Λ tends to y along C_λ is a point W_0 distinct from y. The point W_0 is the intersection of the directrix of the first kind of Wilczynski with the tangent at y to the corresponding curve $C_{-\lambda}$ of Darboux.

Let the homogeneous projective coordinates $y^{(1)}, \cdots, y^{(4)}$ of a general point y on a non-ruled analytic surface S in ordinary space be functions of asymptotic parameters u, v. The functions $y^{(i)}$ are solutions of a system of differential equations, which can be reduced by a suitable transformation to Wilczynski's canonical form

$$(1) \qquad y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + 2a'y_u + gy = 0.$$

The coefficients of these equations are functions of u, v which are connected by three conditions of integrability. Moreover, the coordinates

[1] P. O. Bell, *A study of curved surfaces by means of certain associated ruled surfaces.* Transactions of this Society, vol. 46 (1939), pp. 389–409.

[2] Loc. cit., p. 393.

$y^{(0)}$ are not solutions of any equation of the form $Ay_{uv}+By_u+Cy_v+Dy=0$ whose coefficients are functions of u, v not all zero. This statement implies that the point y_{uv}, whose coordinates are the functions $y_{uv}^{(0)}$, does not lie in the tangent plane to S at y.

An arbitrary one-parameter family F_λ of curves of S is defined by the curvilinear differential equation $dv-\lambda du=0$, where λ is an arbitrary function of u, v. We denote by C_λ the curve of F_λ which passes through y. If u, v be regarded as functions of a single parameter t, as t varies, the point y, whose curvilinear coordinates are u, v, describes a curve of S. This curve will be the curve C_λ if the functions $u=u(t)$, $v=v(t)$ are selected such that for a general value of t

$$\lambda(u, v) = v'/u', \tag{2}$$

where accents indicate differentiation with respect to t.

The curvilinear coordinates of the point Λ are given by $u(t+\overline{\Delta t})$, $v(t+\overline{\Delta t})$. The points U and V are therefore given by $u(t+\overline{\Delta t})$, v and u, $v(t+\overline{\Delta t})$. The general homogeneous coordinates of the points U and V are consequently functions of t, and may be represented by the developments

$$\begin{aligned} U = {} & y + y_u u'\overline{\Delta t} + (y_{uu}u'^2 + y_u u'')\overline{\Delta t}^2/2 \\ & + (y_{uuu}u'^3 + 3u'^2u''y_{uu} + f_1 y_u)\overline{\Delta t}^3/6 \\ & + (y_{uuuu}u'^4 + 6u'^2u''y_{uuu} + f_2 y_{uu} + f_3 u_u)\overline{\Delta t}^4/24 + \cdots, \end{aligned} \tag{3}$$

$$\begin{aligned} V = {} & y + y_v v'\overline{\Delta t} + (y_{vv}v'^2 + y_v v'')\overline{\Delta t}^2/2 \\ & + (y_{vvv}v'^3 + 3v'^2v''y_{vv} + g_1 y_v)\overline{\Delta t}^3/6 \\ & + (y_{vvvv}v'^4 + 6v'^2v''y_{vvv} + g_2 y_{vv} + g_3 y_v)\overline{\Delta t}^4/24 + \cdots \end{aligned} \tag{4}$$

wherein f_i, g_i, $i=1, 2, 3$, represent functions of u, v which for our purpose do not require explicit determination.

By differentiating equations (1) we find that the coefficients of y_{uv} in the expressions for y_{uuu}, y_{vvv}, y_{uuuu}, y_{vvvv} are $-2b$, $-2a'$, $-4b_u$, $-4a_v'$, respectively. The coefficients of y_{uv} in the expressions for the homogeneous coordinates of the points U, V are, therefore,

$$-bu'^2\overline{\Delta t}^3/3 - (b_u u'^4 + 3bu'^2u'')\overline{\Delta t}^4/6 + \cdots,$$

$$-a'v'^3\overline{\Delta t}^3/3 - (a_v' v'^4 + 3a'v'^2v'')\overline{\Delta t}^4/6 + \cdots,$$

respectively. The point W, which is the intersection of the tangent plane to S at y with the line joining U, V has homogeneous coordinates which may be obtained by forming a linear combination of those of U and V which contains no y_{uv} term. Hence, such a combination is

$$(2a'v'^3 + [a'_v v'^4 + 3a'v'^2v'']\overline{\Delta t})U - (2bu'^3 + [b_u u'^4 + 3bu'^2u'']\overline{\Delta t})V.$$

Expanding this we obtain the expression

$$2(a'v'^3 - bu'^3)y + (a'_v v'^4 + 3a'v'^2v'' - b_u u'^4 - 3bu'^2u'')\overline{\Delta t}y + 2(a'v'^3u'y_u - bu'^2v'y_v)\overline{\Delta t} + \text{terms of order } \overline{\Delta t}^2 \tag{5}$$

for the homogeneous coordinates of the point W. If $a'v'^3 - bu'^3 \neq 0$, the limit of W as $\overline{\Delta t}$ tends to zero is, clearly, the point y. Moreover, the tangent to C_v at y has the direction defined by $dv/du = -bu'^2/a'v'^2$. This is the direction of the R_λ-correspondent of the tangent to C_λ at y. This completes the proof for the general case in which λ is not a direction of Segre.

The curve C_λ is a curve of Segre if and only if at each of its points the direction defined by $\lambda = v'/u'$ satisfies the equation $a'v'^3 - bu'^3 = 0$. In this case it is clear from (5) that the limit of W as $\overline{\Delta t}$ tends to zero is a point W_0, distinct from y, whose homogeneous coordinates are given by

$$(3a'v'^2v'' - 3bu'^2u'' + a_v v'^4 - b_u u'^4)y + 2(a'v'^3u'y_u - bu'^2v'y_v), \qquad \text{where } a'v'^3 = bu'^3. \tag{6}$$

If we divide this expression by $a'u'v'^3$, make use of the condition $a'v'^3 = bu'^3$, and make the following substitutions, $v'/u' = \lambda$, $-v'u''/u'^3 = \lambda_u$, $v''/u'v' = \lambda_v$, we obtain the simpler form

$$(3[\lambda_u + \lambda\lambda_v]/\lambda + a'_v\lambda/a' - b_u/b)y + 2(y_u - \lambda y_v), \qquad \text{where } a'\lambda^3 = b, \tag{7}$$

for the coordinates of W_0. It is, clearly, a simple matter to evaluate (7) explicitly for a direction $\lambda = \epsilon(b/a')^{1/3}$, wherein ϵ is a cube root of unity. The result is

$$2y_u - a'_u y/a' - \epsilon(b/a')^{1/3}(2y_v - b_v y/b). \tag{8}$$

This expression is a linear combination of the expressions $2y_u - a_u'y/a'$ and $2y_v - b_v y/b$ for the homogeneous coordinates of the points in which the directrix of the first kind intersects the asymptotic u- and v-tangents to S at y. Moreover, the ratio of the coefficient of y_v to that of y_u is the direction $-\epsilon(b/a')^{1/3}$ of Darboux which corresponds to the direction $\epsilon(b/a')^{1/3}$ of Segre. This completes the demonstration of the theorem.

UNIVERSITY OF KANSAS

ON THE SIMULTANEOUS APPROXIMATION OF TWO REAL NUMBERS[1]

RAPHAEL M. ROBINSON

If $\xi_1, \xi_2, \cdots, \xi_n$ are any real numbers and t is a positive integer, then it is well known that integers $a_1, a_2, \cdots, a_n, b$ can be found, such that $0 < b \leqq t^n$ and

$$|b\xi_k - a_k| < 1/t, \qquad k = 1, 2, \cdots, n.$$

The proof is briefly the following.[2] Consider the t^n+1 points $(r\xi_1, r\xi_2, \cdots, r\xi_n)$, where $r = 0, 1, \cdots, t^n$. Reduce mod 1 to congruent points in the unit cube $(0 \leqq x_1 < 1, \cdots, 0 \leqq x_n < 1)$. If this cube is divided into t^n cubes of edge $1/t$ (including the lower boundaries), then at least one of these small cubes must contain two of the reduced points, say those with $r = r'$ and $r = r''$. With $b = |r' - r''|$ and suitable a's, we evidently satisfy the required inequalities.

For $n = 1$, the inequality can be sharpened to

$$|b\xi - a| \leqq 1/(t+1),$$

b satisfying the condition $0 < b \leqq t$. For if we consider the points $r\xi$ $(r = 0, 1, \cdots, t)$, and mark the points in the interval $0 \leqq x \leqq 1$ which are congruent to them mod 1, we have at least $t+2$ points marked, since corresponding to $r = 0$ we mark both 0 and 1. Some two of the marked points must lie within a distance $1/(t+1)$ from each other, so that the desired conclusion follows. This is the best result, as the example $\xi = 1/(t+1)$ shows.

The present note solves the corresponding problem for $n = 2$. For larger values of n the problem appears more difficult.

THEOREM. *If ξ_1 and ξ_2 are any real numbers, and s is a positive integer, then integers a_1, a_2, b can be found, such that $0 < b \leqq s$, and*

$$|b\xi_k - a_k| \leqq \max\left(\frac{[s^{1/2}]}{s+1}, \frac{1}{[s^{1/2}]+1}\right), \qquad k = 1, 2.$$

For every s, values of ξ_1 and ξ_2 can be found for which the inequalities could not both be satisfied if the equality sign were omitted.

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[1] Presented to the Society, November 23, 1940.

[2] The method used in this proof (*Schubfachprinzip* or "pigeonhole principle") was first used by Dirichlet in connection with a similar problem. We sketch the proof here in order to compare it with the proof of the theorem below, which also uses that method.

The inequalities may also be written

$$|b\xi_k - a_k| \leqq \begin{cases} t/(s+1) & \text{for } t^2-1 \leqq s \leqq t(t+1)-1, \\ 1/(t+1) & \text{for } t(t+1)-1 \leqq s \leqq (t+1)^2-1. \end{cases}$$

It will be noted that in some intervals the bound does not decrease as s increases.

We show first that the theorem is the best possible. We shall think of the inequalities in the form just given. If $s<(t+1)^2$, then it is evident that $\xi_1=1/(t+1)$, $\xi_2=1/(t+1)^2$ are a pair of real numbers which cannot be approximated simultaneously with an error less than $1/(t+1)$; this settles the second case. For the first case, consider the pair of real numbers $\xi_1=1/(s+1)$, $\xi_2=t/(s+1)$. We are to show that not both errors can be made less than $t/(s+1)$. We note first that $b\xi_1$ and $b\xi_2$ differ from integers by the same amount as $(s+1-b)\xi_1$ and $(s+1-b)\xi_2$; hence we may suppose that $b\leqq(s+1)/2$, and therefore $0<b\xi_1\leqq 1/2$. In order to make $|b\xi_1-a_1|<t/(s+1)$, we must have $0<b<t$. Then $0<b\xi_2<1$. Since $b\xi_2\geqq\xi_2=t/(s+1)$ and $1-b\xi_2\geqq 1-(t-1)\xi_2=1-(t-1)t/(s+1)\geqq t/(s+1)$, we see that the inequality $|b\xi_2-a_2|<t/(s+1)$ cannot be satisfied.

The theorem evidently follows from the lemma below, by putting $t=[s^{1/2}]$.

LEMMA. *Let s and t be positive integers with $s\geqq t$. If ξ_1 and ξ_2 are any real numbers, then integers a_1, a_2, b can be found, such that $0<b\leqq s$, and*

$$|b\xi_1 - a_1| \leqq t/(s+1), \qquad |b\xi_2 - a_2| \leqq 1/(t+1).$$

PROOF. Consider the points $(r\xi_1, r\xi_2)$ with $r=0, 1, \cdots, s$. Mark all the points congruent to these mod 1 which fall in the rectangle $0\leqq x_1\leqq t$, $0\leqq x_2<1$. There are $(s+1)t$ points to be marked with $x_1<t$; and in addition, the point $(t, 0)$ is marked, corresponding to $r=0$. If we divide our rectangle into $s+1$ rectangles of width $t/(s+1)$ (closed except at the top) by means of vertical lines, then at least one of them contains more than t points, all corresponding to different values of r. The corresponding values of x_2 are $t+1$ or more numbers, some two of which differ mod 1 by not more than $1/(t+1)$. Thus we find two points $(r'\xi_1, r'\xi_2)$ and $(r''\xi_1, r''\xi_2)$, whose horizontal distance mod 1 does not exceed $t/(s+1)$ and whose vertical distance mod 1 does not exceed $1/(t+1)$. Putting $b=|r'-r''|$ gives the required result.

UNIVERSITY OF CALIFORNIA

NOTE ON A THEOREM ON QUADRATIC RESIDUES

KAI-LAI CHUNG

In this note we shall give a short proof of a known result:

THEOREM. *For every prime* $p \equiv 3 \pmod 4$ *there are more quadratic residues* mod p *between* 0 *and* $p/2$ *than there are between* $p/2$ *and* p.

An equivalent statement of this theorem is as follows (see E. Landau, *Vorlesungen über Zahlentheorie*, vol. 1, p. 129):

Für $p \equiv 3 \pmod 4$ *haben mehr unter den Zahlen* $1^2, 2^2, \cdots, (p-1)^2/4$ *ihren Divisionsrest* mod p *unter* $p/2$ *als über* $p/2$.

For proof we shall use Fourier series with one of its applications, namely Gaussian sums.

Write $s^2 = qp + r$, $0 \leqq r < p$, so that

$$\left[\frac{s^2}{p}\right] = q.$$

It is evident that we have

$$\left[\frac{2s^2}{p}\right] - 2\left[\frac{s^2}{p}\right] = \begin{cases} 0 & \text{if } r < p/2; \\ 1 & \text{if } r > p/2. \end{cases}$$

Therefore we have to prove that $\sum_{s=1}^{(p-1)/2}([2s^2/p] - 2[s^2/p]) < (p-1)/4$, or $\leqq (p-1)/4$ since $p \equiv 3 \pmod 4$.

By a well known expansion in Fourier series, we have

$$x - [x] - \tfrac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi},$$

so that

$$[x] = x - \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi}.$$

Substituting, we get

$$\begin{aligned} \left[\frac{2s^2}{p}\right] - 2\left[\frac{s^2}{p}\right] &= \frac{2s^2}{p} - \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin (4n\pi s^2/p)}{n\pi} \\ &\quad - 2\left\{\frac{s^2}{p} - \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin (2n\pi s^2/p)}{n\pi}\right\} \\ &= \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}\left\{\sin \frac{4n\pi s^2}{p} - 2 \sin \frac{2n\pi s^2}{p}\right\}; \end{aligned}$$

$$\sum_{s=1}^{(p-1)/2}\left(\left[\frac{2s^2}{p}\right]-2\left[\frac{s^2}{p}\right]\right)$$
$$=\frac{p-1}{4}+\sum_{n=1}^{\infty}\frac{1}{n\pi}\left\{\sum_{s=1}^{(p-1)/2}\left(\sin\frac{4n\pi s^2}{p}-2\sin\frac{2n\pi s^2}{p}\right)\right\}.$$

Therefore we have to prove that

$$\sum_{n=1}^{\infty}\frac{1}{n\pi}\sum_{s=1}^{(p-1)/2}\sin\frac{4n\pi s^2}{p}\leqq\sum_{n=1}^{\infty}\frac{2}{n\pi}\sum_{s=1}^{(p-1)/2}\sin\frac{2n\pi s^2}{p}.$$

Now we have by the results on Gaussian sums,

$$\text{if } \left(\frac{2n}{p}\right)=1,\quad \sum_{s=1}^{(p-1)/2}e^{2n(2\pi is^2/p)}=\sum_{(r/p)=1}e^{2\pi ir/p}=+\frac{ip^{1/2}}{2};$$
$$\text{if } \left(\frac{2n}{p}\right)=-1,\quad \sum_{s=1}^{(p-1)/2}e^{2n(2\pi is^2/p)}=\sum_{(r/p)=1}e^{-2\pi ir/p}=-\frac{ip^{1/2}}{2};$$
$$\text{if } \left(\frac{2n}{p}\right)=0,\quad \sum_{s=1}^{(p-1)/2}e^{2n(2\pi is^2/\nu)}=\frac{p-1}{2}.$$

Taking imaginary parts, we obtain

$$\sum_{s=1}^{(p-1)/2}\sin\frac{4n\pi s^2}{p}=\left(\frac{2n}{p}\right)\frac{p^{1/2}}{2}.$$

Similarly,

$$\sum_{s=1}^{(p-1)/2}\sin\frac{2n\pi s^2}{p}=\left(\frac{n}{p}\right)\frac{p^{1/2}}{2}.$$

Therefore we have to prove that

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{2n}{p}\right)\frac{p^{1/2}}{2}\leqq\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{n}{p}\right)\frac{p^{1/2}}{2};$$

that is,

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{2n}{p}\right)\leqq 2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{n}{p}\right).$$

This is equivalent to the relation

$$\sum_{n=1}^{\infty}\frac{-1}{n}\left(\frac{n}{p}\right)\leqq 2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{n}{p}\right),\quad \text{if } p\equiv 3 \pmod 8;$$

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{n}{p}\right) \leqq 2\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{n}{p}\right), \quad \text{if } p \equiv 7 \pmod 8.$$

Thus in each case we have to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{n}{p}\right) \geqq 0.$$

Now Landau would call this last result trivial. But for the sake of completeness we give its proof here; we have in fact, for $s>1$,

$$\left\{\sum_{n=1}^{\infty} \frac{1}{n^s}\left(\frac{n}{p}\right)\right\} \prod_{p_1}\left(1 - \frac{1}{p_1^s}\left(\frac{p_1}{p}\right)\right) = 1,$$

where p_1 runs through the sequence of primes. The series being uniformly convergent for $s \geqq 1$ its sum is continuous at $s=1$. Hence the result follows.

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